1. (a) Let $n$ be a positive integer. Using Cauchy’s integral formula, calculate the integral
\[ \oint_C \left( z - \frac{1}{z} \right)^n \frac{dz}{z}, \]
where $C$ is the unit circle in $\mathbb{C}$.

**Solution.** Let $f(z) = (z^2 - 1)^n$. Since $f$ is holomorphic and
\[ \oint_C \left( z - \frac{1}{z} \right)^n \frac{dz}{z} = \oint_C \frac{(z^2 - 1)^n}{z^{n+1}} \, dz, \]
the Cauchy Integral Formula tells us that
\[ \oint_C \left( z - \frac{1}{z} \right)^n \frac{dz}{z} = \frac{2\pi i f^{(n)}(0)}{n!}. \]

Now we expand $f$ taking derivatives:
\[ f(z) = (z^2 - 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k z^{2n-2k}, \]
so that
\[ f^{(n)}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (-1)^k (2n-2k) \cdots (n-2k) z^{n-2k}. \]

This means
\[ f^{(n)}(0) = \begin{cases} \binom{n}{n/2} (-1)^{n/2} n! & n \text{ is even}, \\ 0 & n \text{ is odd}. \end{cases} \]

Hence
\[ \oint_C \left( z - \frac{1}{z} \right)^n \frac{dz}{z} = \begin{cases} 2\pi i \binom{n}{n/2} (-1)^{n/2} & n \text{ is even}, \\ 0 & n \text{ is odd}. \end{cases} \]

(b) By using the substitution $z \mapsto e^{it}$ in the integral above, evaluate
\[ \int_0^{2\pi} \sin^n z \, dz. \]
Solution. The desired substitution in the above integral yields

\[
\oint_C \left( z - \frac{1}{z} \right)^n \frac{dz}{z} = \int_0^{2\pi} (e^{it} - e^{-it})^n \frac{ie^i t \, dt}{e^t}
\]

\[
= i \cdot (2i)^n \int_0^{2\pi} \sin^n t \, dt.
\]

By part (a) we easily compute that

\[
\int_0^{2\pi} \sin^n t \, dt = \begin{cases} \frac{\pi}{2^n} \binom{n}{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}
\]

\[\square\]

2. Let \( \tau \) be a complex number that is not real. Let \( f(z) \) be a holomorphic function such that \( f(z + 1) = f(z) \) and \( f(z + \tau) = f(z) \). Prove that \( f \) is constant.

Solution. Ah...elliptic functions. The two conditions on \( f \) tell us that this function is “doubly periodic”. Put another way, the values of \( f \) in \( \mathbb{C} \) are determined by the values \( f \) takes on the parallelogram spanned by the origin, 1, \( \tau \) and \( 1 + \tau \). (Note this is a non-degenrate parallelogram since \( \tau \) is not real.) But the parallelogram is a compact region in \( \mathbb{C} \). Hence \( f \) achieves a maximum \( M \) in this region and by translation \( f(z) \leq M \) for all \( z \in \mathbb{C} \). Since \( f \) is entire (i.e., holomorphic in all of \( \mathbb{C} \)) and bounded, Liouville’s theorem tells us \( f \) is constant. \[\square\]

3. Let \( P(z) = a_0 + a_1 z + \cdots + a_n z^n \), where \( a_n \neq 0 \). Show there exist \( n \) complex numbers \( \alpha_1, \ldots, \alpha_n \), possibly not distinct, such that

\[
P(z) = a_n (z - \alpha_1) \cdots (z - \alpha_n).
\]

Solution. We use induction on \( n \). The case \( n = 0 \) is clear. Suppose the claim holds for all polynomials of degree less than or equal to \( n - 1 \). Let \( P(z) \) be a polynomial of degree \( n \) as above. Then by the fundamental theorem of algebra \( P(z) \) has a root. Call this root \( \alpha_n \). Then the division algorithm (which holds in \( \mathbb{C} \)) tells us that

\[
P(z) = (z - \alpha_n)Q(z) + R(z),
\]

where \( R \) in this case is constant. Since \( P(\alpha_n) = 0 \), it follows that \( R(z) = R(\alpha_n) = 0 \). Note that the leading coefficient of \( Q \) is still \( a_n \) and that \( Q \) has degree \( n - 1 \). By inductive hypothesis there exist \( n - 1 \) possibly non-distinct numbers \( \alpha_1, \ldots, \alpha_{n-1} \) such that

\[
Q(z) = a_n (z - \alpha_1) \cdots (z - \alpha_{n-1}).
\]

Whence

\[
P(z) = a_n (z - \alpha_1) \cdots (z - \alpha_n).
\]

\[\square\]
4. Let $C$ be the circle $|z| = 2$ in $\mathbb{C}$. Evaluate the integral

$$\frac{1}{2\pi i} \oint_C f(z) \, dz$$

for the following functions $f(z)$. Here $k \in \mathbb{N}$.

**Solution.** I’ll do a couple of the problematic integrals. I myself messed up in part (p). If you want your point back send me an email and I’ll gladly give it to you.

(m) $(z - \sin z)/(z^2 \sin z)$

**Solution.** Zero is the only singularity of the function inside $C$. Now note that $\sin z$ has a simple zero at 0. Hence the residue at 0 of this function is just

$$\frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left( \frac{z(z - \sin z)}{\sin z} \right) = 0.$$

Hence

$$\frac{1}{2\pi i} \oint_C f(z) \, dz = \sum \text{Res} = 0.$$

(p) $e^{1/z}/(1 - z)$

**Solution.** A problem like this should begin to ring “inversion” in your mind. (It didn’t in mine at first...) The trick is to use the substitution $z \mapsto -1/t$. This takes the unit disk into the outside world and viceversa. The minus sign is there to preserve the orientation (counterclockwise) of the image of the contour $C$. Under this map $C$ maps to $|z| = 1/2 =: C'$. Hence

$$\frac{1}{2\pi i} \oint_C \frac{e^{1/z}}{1 - z} \, dz = \frac{1}{2\pi i} \oint_{C'} \frac{e^{-t}}{1 - 1/t} \cdot \frac{-1}{t^2} \, dt = \frac{1}{2\pi i} \oint_{C'} \frac{e^{-t}}{t(1 - t)} \, dt.$$

The only singularity inside $C'$ is $t = 0$. The residue at 0 is just

$$\lim_{t \to 0} \frac{t e^{-t}}{t(1 - t)} = -1.$$

Hence

$$\frac{1}{2\pi i} \oint_C \frac{e^{1/z}}{1 - z} \, dz = \frac{1}{2\pi i} \oint_{C'} \frac{e^{-t}}{t(1 - t)} \, dt = \sum \text{Res} = -1.$$
(u) \( \tan z/z^2 \)

Solution. Note that \( f(z) = \sin z/(z^2 \cos z) \). The singularities of this function inside \( C \) lie at \(-\pi/2\), 0, and \( \pi/2 \). The cosine function has simple zeroes at \(-\pi/2\) and \( \pi/2 \). The function has a pole of order 2 at 0, hence the residue at the origin is

\[
\lim_{z \to 0} \frac{d}{dz} \left( \frac{z^2 \sin z}{z^2 \cos z} \right) = 1.
\]

The residues at \(-\pi/2\) and \( \pi/2 \) are, respectively,

\[
\lim_{z \to -\pi/2} \frac{(z - \pi/2) \sin z}{z^2 \cos z} \quad \text{and} \quad \lim_{z \to \pi/2} \frac{(z + \pi/2) \sin z}{z^2 \cos z}.
\]

Using L'Hôpital's rule we see that both these expressions evaluate to \(-4/\pi^2\). Hence

\[
\frac{1}{2\pi i} \oint_C \frac{\tan z}{z^2} \, dz = \sum \text{Res} = 1 - \frac{4}{\pi^2} - \frac{4}{\pi^2} = 1 - \frac{8}{\pi^2}.
\]

\(\square\)