Problem Set 3 Solution Set

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1. Let $f(z) = e^z$. Let $a$ be a positive real number, and let $C$ be the rectangle with vertices $0, a, a + 2\pi i$ and $2\pi i$. Explicitly compute the integral

$$\oint_C f(z) \, dz$$

without using Cauchy's theorem, and verify that Cauchy's theorem applies in this case.

![Diagram of rectangle with vertices 0, a, a + 2\pi i, and 2\pi i]

**Solution.** Colloquially, we write

$$\oint_C = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} .$$

Now we compute the four integrals individually:

$$\int_{\gamma_1} e^z \, dz = \int_0^a e^t \, dt = e^a - 1 .$$

$$\int_{\gamma_2} e^z \, dz = \int_0^1 e^{(1-t)+(a+2\pi i)t} 2\pi i \, dt = e^a e^{2\pi i t}\bigg|_0^1 = 0 .$$

$$\int_{\gamma_3} e^z \, dz = \int_0^1 e^{(a+2\pi i)(1-t)+2\pi it} (-a) \, dt = e^a \int_0^1 e^{-at} (-a) \, dt = e^a e^{-at}\bigg|_0^1 = 1 - e^a .$$

$$\int_{\gamma_4} e^z \, dz = \int_0^1 e^{2\pi i (1-t)} (-2\pi i) \, dt = e^{-2\pi it}\bigg|_0^1 = 0 .$$

Hence

$$\oint_C f(z) \, dz = (e^a - 1) + 0 + 0 + (1 - e^a) = 0 .$$

This verifies Cauchy's Integral Theorem since $f$ is analytic inside the region defined by $C$. □
2. Let \( \gamma \) be the semicircular arc from 1 to \(-1\) in the upper half plane. Use the \( ML \)-inequality to prove that
\[
\left| \int_{\gamma} \frac{e^z}{z} \, dz \right| \leq \pi e.
\]

Solution. The \( ML \)-inequality tells us in this case that
\[
\left| \int_{\gamma} \frac{e^z}{z} \, dz \right| \leq \max_{\gamma} \left| \frac{e^z}{z} \right| \cdot L(\gamma),
\]
where \( L(\gamma) \) is the length of \( \gamma \). We know from Kindergarten that \( L(\gamma) = \pi \). Now,
\[
\max_{\gamma} \left| \frac{e^z}{z} \right| = \max_{\gamma} \frac{|e^z|}{|z|} = \max_{\gamma} |e^z| = \max_{\gamma} |e^{x+i\beta}| = \max e^{x} = e.
\]
This gives the desired result. \( \square \)

3. Let \( R \) be the region \( \mathbb{C} \setminus \{[0, \infty]\} \). Let \( f(z) = \sqrt{z} \), considered as a holomorphic function on \( R \), and such that \( f(-1) = i \).

(a) Let \( \epsilon \) be a small real number, and let \( \gamma_\epsilon \) be the path along the unit circle given explicitly by \( e^{it} \) for \( t \in [\epsilon, 2\pi - \epsilon] \). Compute the integral
\[
I_\epsilon = \int_{\gamma_\epsilon} \sqrt{z} \, dz.
\]

Solution.
\[
I_\epsilon = \int_{\epsilon}^{2\pi-\epsilon} e^{it/2} \cdot ie^{it} \, dt = i \int_{\epsilon}^{2\pi-\epsilon} e^{3it/2} \, dt
\]
\[
= (2/3) e^{3it/2} \bigg|_{\epsilon}^{2\pi-\epsilon} = (2/3) (e^{3\pi i - 3\epsilon i/2} - e^{3\epsilon i/2})
\]
\[
= (2/3) (-e^{-3\epsilon i/2} - e^{3\epsilon i/2}) = -\frac{4}{3} \cos \left( \frac{3\epsilon}{2} \right).
\]
\( \square \)

(b) Let \( I \) be the limit of the integral as \( \epsilon \) approaches 0. Compute \( I \), and explain why the fact that \( I \neq 0 \) does not contradict Cauchy’s theorem.

Solution.
\[
I = \lim_{\epsilon \to 0} I_\epsilon = \lim_{\epsilon \to 0} -\frac{4}{3} \cos \left( \frac{3\epsilon}{2} \right) = -\frac{4}{3}.
\]
The fact that \( I \neq 0 \) does not contradict Cauchy’s theorem because \( f \) is multivalued (and hence not analytic) on any simple curve that encloses the origin. Another way to say this is the branch cut we introduced to make \( f \) analytic makes the path for \( I \) not closed, so that Cauchy’s theorem does not apply. \( \square \)
(c) Compute the real integral

\[ J = \int_0^1 \sqrt{x} \, dx. \]

Proof. A Math 1a no-brainer:

\[ \int_0^1 \sqrt{x} \, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}. \]

(d) Prove directly without computing \( I \) or \( J \) that \( I + 2J = 0 \).

\[ \begin{tikzpicture}
\draw (0,0) circle (2cm);
\draw[->] (0,0) -- (3,0);
\draw[->] (0,0) -- (0,3);
\node at (0,0) {O};
\node at (3,0) {A};
\node at (0,3) {B};
\node at (2,2) {$\gamma_1$};
\node at (1,1) {$\gamma_2$};
\node at (-1,1) {$\gamma_3$};
\end{tikzpicture} \]

Solution. Let \( A = e^{ie} \) and \( B = e^{i(2\pi - e)} \) in the figure above. Since \( \sqrt{0} = 0 \) the function \( f \) is analytic in the region area enclosed by \( \gamma_1 + \gamma_2 + \gamma_3 \). By Cauchy’s integral formula,

\[ \int_{\gamma_1 + \gamma_2 + \gamma_3} \sqrt{z} \, dz = 0 \]

However,

\[ \lim_{\epsilon \to 0} \int_{\gamma_1 + \gamma_2 + \gamma_3} = \lim_{\epsilon \to 0} \left( \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_1} \right) = \lim_{\epsilon \to 0} \int_{\gamma_1} + \lim_{\epsilon \to 0} \int_{\gamma_2} + \lim_{\epsilon \to 0} \int_{\gamma_3}. \]

Now note that \( \lim_{\epsilon \to 0} \int_{\gamma_1} = I \) and \( \lim_{\epsilon \to 0} \int_{\gamma_2} = J \). It remains to see what \( \lim_{\epsilon \to 0} \int_{\gamma_3} \) evaluates to. Since \( \gamma_3 \) has opposite orientation from \( \gamma_2 \) this integral gains a minus sign with respect to \( J \). However, as \( \epsilon \to 0 \) the values of \( \sqrt{z} \) approach the negative real values of the real square root function. So we pick up a second minus sign. We conclude \( \lim_{\epsilon \to 0} \int_{\gamma_3} = J \). Hence \( I + 2J = 0. \)
4. If $R$ is a simply connected region with boundary $C$, prove that

$$A = \frac{1}{2i} \oint_C \overline{z} \, dz,$$

where $A$ is the area of $R$.

**Solution.** We apply Green’s theorem. Write $\overline{z} = x - iy$, $dz = dx + idy$. All partial derivatives being continuous, we compute

$$\oint_C \overline{z} \, dz = \oint_C x \, dx + y \, dy + i \oint_C -y \, dx + x \, dy$$

$$= \iint_R (0 - 0) \, dA + i \iint_R (1 - (-1)) \, dA$$

$$= 2iA,$$

from which the desired equality follows immediately. \qed