Solutions to PS 9 (Math 121)

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Question 1: 6.1/20

Prove the Polar Identities:

1. $||x+y||^2+||x-y||^2 = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle = \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle + 2\langle y, x\rangle - \langle x, x\rangle - \langle y, y\rangle = 4\langle x, y\rangle$.
2. $||x+y||^2+i||x+iy||^2-||x-y||^2-i||x-iy||^2 = (||x+y||^2-||x-y||^2)+i(||x+iy||^2-||x-iy||^2) = 4\langle x, y\rangle$.

Question 2: 6.1/24d

Show that $||(a, b)|| = \max\{|a|, |b|\}$ is a norm.

1. Positive-definitness: clearly by definition it is positive. Also the only way $\max\{|a|, |b|\} = 0$ is when $(a, b) = (0, 0)$.
2. $||c(a, b)|| = ||(ca, cb)|| = \max\{|ca|, |cb|\} = |c| \max\{|a|, |b|\} = |c|||(a, b)||$.
3. Note that $|a + c| < |a| + |c|$ and $|b + d| < |b| + |d|$ and so the inequality holds.

Question 3: 6.1/25

If the inner product was defined using the polar identity and the definition of the norm from 24(d) then it would not be linear. It would be sufficient to show this on the pairs $x = (1, 0)$ and $y = (2, 2)$:

$$\langle (1, 0), (2, 2)\rangle = \frac{1}{4}(9 - 4) = \frac{5}{4} \neq 2\langle (1, 0), (1, 1)\rangle = \frac{1}{2}(4 - 1) = \frac{3}{2}$$

Question 4: 6.2/2(i)

I am not including the solution since it is just computation. However, if you find this any problems with this question do not hesitate to contact me.

Question 5: 6.2/13

Let S and S_0 be subsets of a finite dimensional vector space V.

1. If suppose $x \in S^\perp$ then this means that $\langle x, y\rangle$ for all $y \in S$ and so $y \in S_0$. Therefore $x \in S^\perp$ implies that $x \in S_0^\perp$ and hence $S^\perp \subset S_0^\perp$.
2. Let $x \in S$. Therefore $\langle x, y\rangle = 0$ for any $y \in S^\perp$ and so $x \in (S^\perp)^\perp$, which implies $S \subset (S^\perp)^\perp$. Moreover, since all the inner products are linear that the statement holds for the span as well.

3. On side of the equality holds be the previous part so it is sufficient to show that $(W^\perp)^\perp \subset W$. Suppose $x \notin W$. Then this means (thanks to exercise 6) there exist $y \in W^\perp$ such that $\langle x, y\rangle \neq 0$ which in turn means that $x \notin (W^\perp)^\perp$. Therefore by contrapositive we are done.

4. We need to show to things: 1) $W \cap W^\perp = 0$. But the only vector that is perpendicular to itself is (by positive definitness) the zero vector so we are fine. 2) The theorem 6.6 clearly shows that $V = W + W^\perp$.

Question 6: 6.2/14

1. $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$. Suppose $x \in (W_1 + W_2)^\perp$, that means in particular that x must be in both $W_1^\perp$ and $W_2^\perp$. On the other hand if x is in both $W_1^\perp$ and $W_2^\perp$, then by linearity of inner products it will be in $(W_1 + W_2)^\perp$.

2. $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$. Suppose $x \in (W_1 \cap W_2)^\perp$ and so it is perpendicular to everything that is in both W_1 and W_2, then quite naturally it will be in $W_1^\perp + W_2^\perp$ as this is a superset of $W_1^\perp$ and $W_2^\perp$. On the other hand if $x \in W_1^\perp + W_2^\perp$ then this means that it is a combination of two vectors where each one is perpendicular to W_1 respectively W_2. But this means they will be perpendicular to the intersection of W_1 and W_2 since it only contains things that are in both.

Question 7: 6.3/9

Let $V = W + W^\perp$ and let T be the projection on W along $W^\perp$. Let us start with $\langle Tx, y\rangle = \langle T(x_1 + x_2), (y_1 + y_2) = \langle x_1, (y_1 + y_2)\rangle = \langle x_1, y_1\rangle = \langle (x_1 + x_2), y_1\rangle = \langle x, Ty\rangle$. By uniqueness of the definition of the adjoint then $T = T^*$.

Question 8: 6.3/12

Let T be a linear operator on a vectorspace V. Then:

1. Suppose $x \in N(T)$. Therefore $Tx = 0$, hence $0 = \langle Tx, y\rangle = \langle x, T^*y\rangle$. Hence $x \in R(T^*)^\perp$. This argument works both ways and so we are done.

2. Using the hint we know that $N(T)^\perp = (R(T^*)^\perp)^\perp = R(T^*)$. Done.