Solutions to PS 8 (Math 121)

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Question 1: 7.1/2(a)

We are to find the basis of the generalazied eigenvectors and the corresponding Jordan form of matrix A:

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

This matrix gives us a double eigenvalue of $\lambda = 2$. Hence we need to find all the vectors such that:

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

This is satisfied by $a(1,1)$. Therefore we need to search for another generalized eigenvector. So we need to find a vector such that $(A-2I)^2 v = 0$ but $(A-2I)v \neq 0$. The solution to the first equation is any vector since $(A-2I)^2 = 0$, therefore we just need to find a vector such that $(A-2I) \neq 0$, hence anything linearly independent from $a(1,1)$, pick $v = (-1,0)$, so that $(A-2I)v = (1,1)$. Then the Jordan form is:

$$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Question 2: 7.1/3(a)

First, by action on the standard basis we can establish that:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

The only (triple) eigenvalue is again $\lambda = 2$. We can see that the rank$(A-2I) = 2$ and so there must be a three-cycle of generalized eigenvectors, such that $(A-2I)^3 v = 0$ but $(A-2I)^2 v \neq 0$. But we know that $(A-2I)^3 = 0$ so we just need to find a vector such that $(A-2I)^2 v \neq 0$. Since x^2 does the job then we now that our basis is $\{x^2, (T-2I)x^2, (T-2I)^2 x^2\} = \{x^2, -2x, 1\}$. Then the Jordan form will be:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Question 3: 7.1/7

Let T be a linear operator on a finite dimensional V.

1. Show that $N(T^k) \subset N(T^{k+1})$
   Suppose $v \in N(T^k)$, therefore $T^k v = 0$ and so $TT^k v = T0 = 0$, hence $v \in N(T^{k+1})$. Therefore $N(T^k) \subset N(T^{k+1})$ which implies the required chain of subsets.

2. Show that if at some point $\text{rank}(T^n) = \text{rank}(T^{n+1})$ then in fact $\text{rank}(T^n) = \text{rank}(T^{n+k})$ for any $k > 0$.
   Since V is finite dimensional we can use rank-nullity theorem to establish that $\text{null}(T^n) = \text{null}(T^{n+1})$ and also thanks to 1) this means $N(T^n) = N(T^{n+1})$. This in turn means that $T^n v = 0 \Leftrightarrow T^{n+1}v = 0$ for $\forall v \in V$. If this is true for all $v \in V$ then it also must be true or all $v = Tw$ with $\forall w \in V$ and so this means that $T^n v = 0 \Leftrightarrow T^{n+1}v, \forall v \in V$ is equivalent to $T^{n+1}w = 0 \Leftrightarrow T^{n+2}w, \forall w \in V$. This will mean the nullities are equal which in turn means the ranks will be equal from now on as well.

3. The previous part proves this result as well.

4. We are given that $\text{rank}((T - \lambda I)^n) = \text{rank}((T - \lambda I)^{n+1})$, also we know that $K_\lambda = N((T - \lambda I)^m)$, where m is the multiplicity of λ. By trichotomy, there are only three possible cases: $n < m$, $m < n$ or $n = m$:

   (a) $n < m$ - That means that $\exists v$ such that $v \in N((T - \lambda I)^m)$ but $v \notin N((T - \lambda I)^n)$, which is a contradiction to c).

   (b) $m < n$ - That means that $\exists v$ such that $v \in N((T - \lambda I)^m)$ but $v \notin N((T - \lambda I)^n)$, which is a contradiction to a).

   (c) therefore $m = n$ must hold.

5. If $\text{rank}(T - \lambda I) = \text{rank}((T - \lambda I)^2)$, then this means that $K_{\lambda_i} = N(T - \lambda_i I) = E_{\lambda_i}$. This implies that the transformation is diagonalizable.

6.

Question 4: 6.1/4

1. We just need to check part b and c:

   (a) Done in the book.

   (b) Check that $\langle cx, y\rangle = c\langle x, y\rangle$.
   $\langle cA, B\rangle = \text{tr}(B^*(cA)) = c\text{tr}(B^*A) = c\langle x, y\rangle$.

   (c) Check that $\langle x, y\rangle = \overline{\langle x, y\rangle}$.
   $\langle x, y\rangle = \text{tr}(B^*A) = \text{tr}((\overline{B})^t A) = \text{tr}(A^t\overline{B}) = \overline{\text{tr}}(A^*B) = \overline{\langle x, y\rangle}$.

   (d) Done in the book.

2. Compute the following:

   (a) $||A|| = \sqrt{\langle A, A\rangle} = \sqrt{1+1+9+5} = 4$

   (b) $||B|| = \sqrt{\langle A, A\rangle} = \sqrt{1+1+2} = 2$

   (c) $\langle A, B\rangle = 1 + i + 3i + 1 = 2 + 4i$

Question 5: 6.1/8

We are supposed to give reasons why the following are not inner products on their respective spaces.

1. This is not an innerproduct because it is skew symmetric.

2. This product does not factor scalars since $\text{cA} + \text{B} \neq c\text{tr}(A + B)$.

3. This product is not definite. Take f to be a constant function then $\langle f, f\rangle = 0$ even though f is not a zero function.

Question 6: 6.1/11

1. First we will prove this equality: $||x+y||^2 + ||x-y||^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle = 2\langle x,x\rangle + 2\langle y,y\rangle + \langle x,y\rangle - \langle x,y\rangle + \langle y,x\rangle - \langle y,x\rangle = 2||x||^2 + 2||y||^2$

2. Now, we will explain the result. First note that $x+y$ and $x-y$ are the diagonals of the parallelogram, hence the sum of the length squared of the sides is half the sum of the length squared of the diagonals.

Question 7: 6.1/13

let us check whether the new inner product satisfies our requirements.

1. $\langle x+z, y\rangle = \langle x+z, y\rangle_1 + \langle x+z, y\rangle_2 = \langle x,y\rangle_1 + \langle z,y\rangle_1 + \langle x,y\rangle_2 + \langle z,y\rangle_2 = \langle x,y\rangle + \langle z,y\rangle$

2. $\langle cx, y\rangle = \langle cx, y\rangle_1 + \langle cx, y\rangle_2 = c(\langle x,y\rangle_1 + \langle x,y\rangle_2) = c\langle x,y\rangle$

3. $\langle x,y\rangle = \langle x,y\rangle_1 + \langle x,y\rangle_2 = \overline{\langle y,x\rangle_1} + \overline{\langle y,x\rangle_2} = \overline{\langle y,x\rangle_1 + \langle y,x\rangle_2} = \overline{\langle y,x\rangle}$

4. $\langle x,x\rangle = \langle x,x\rangle_1 + \langle x,x\rangle_2$ and hence this is zero iff $x=0$ and it is never negative.