

Solutions to PS 7 (Math 121)

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Question 1: 5.2/12

1. Suppose that $v \in E_\lambda$, then $T(v) = \lambda v$ and hence by taking the inverse of both sides we get: $v = \lambda T^{-1}(v)$ and so $v \in E_{\lambda^{-1}}$. Hence the two eigenspaces are equal.
2. Suppose that $A = [T]$ is diagonalizable then there exist $D = Q^{-1}AQ$ such that D is diagonal. Then by taking an inverse of each side we get $D^{-1} = Q^{-1}A^{-1}Q$. Note, that since D is diagonal it is invertible and our last steps is hence justified.

Question 2: 5.2/14a

Find a solution to the system of differential equations: $x' = x + y$ and $y' = 3x - y$.

We need to diagonalize the matrix $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$, which gives us $D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ and $Q = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$. And so our new system looks like $p' = -2p$ and $q' = 2q$, therefore $p = Ce^{-2t}$ and $q = De^{2t}$. Now we only need to convert to the old basis where $p = -x+3y$ and $q = x+y$ and so $y = \frac{1}{4}\{Ce^{-2t} + De^{2t}\}$, whereas $x = -\frac{1}{4}\{Ce^{-2t} - 3De^{2t}\}$.

Question 3: 5.2/17

Two operators T and U are simultaneously diagonalizable iff there exists a basis β such that both $[T]_\beta$ and $[U]_\beta$ are diagonal. Similarly two matrices A, B are simultaneously diagonalizable if there exists an invertible Q such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.

1. Show that if T and U are simultaneously diagonalizable then the matrices $[T]_\beta$ and $[U]_\beta$ are also simultaneously diagonalizable.
If both T and U are both simultaneously diagonalizable then there exists a basis γ such that both $[T]_\gamma$ and $[U]_\gamma$ are diagonal. But this means that $Q = [I]_\beta^\gamma$ is matrix such that both $Q^{-1}[T]_\beta Q = [T]_\gamma$ and $Q^{-1}[U]_\beta Q = [U]_\gamma$ are diagonal.
2. Show that if A and B are two simultaneously diagonalizable matrices then L_A and L_B are simultaneously diagonalizable linear transformations.
If A and B are two simultaneously diagonalizable matrices, then this means that there exists an invertible Q such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal. This means that Q can be understood as a change of basis from the standard basis to a basis β . Then $Q^{-1}AQ = [L_A]_\beta$ and $Q^{-1}BQ = [L_B]_\beta$ are both diagonal.

Question 4: 5.4/3

Show that the following subspaces are T -invariant for any linear operator T on V :

1. $\{0\}$ is invariant simply because $T(0) = 0 \in \{0\}$

2. V is invariant since T is a linear operator on V and hence $T(V) \subset V$.
3. $N(T)$ is invariant since $T(N(T)) = 0 \in N(T)$
4. $R(T)$ is invariant as $T(V) = R(T)$ and hence $T(R(T)) \subset T(V) = R(T)$.
5. E_λ is invariant since $T(E_\lambda) = \lambda E_\lambda = E_\lambda$.

Question 5: 5.4/25a

Let us first show that the eigenspace of T for eigenvalue λ is invariant under U . Suppose that $v \in E_\lambda$. Then $TUv = UTv = U(\lambda v) = \lambda U(v)$ and so if v is an eigenvector of T then $U(v)$ is also an eigenvector with the same eigenvalue. Therefore E_λ is U -invariant.

Now by exercise 24 any restriction of a diagonalizable operator on a non-trivial invariant subspace is diagonalizable. Since we know that E_λ is non-trivial, then we know that the restriction U_{E_λ} is a diagonalizable and moreover no matter what basis we pick to do diagonalization of this the restriction, T_{E_λ} will also be diagonal. (This is because any basis for E_λ will give us a diagonal submatrix for T) Hence by picking all the eigenspaces of T we can find a basis for V such that in which both U and T are diagonal.

Question 6: 5.4/33

Let T be linear operator and W_1, \dots, W_k are T -invariant subspaces of V . Show that $W_1 + W_2 + \dots + W_k$ is also a T -invariant subspace.

We will show this for two T -invariant subspaces and the required result will follow by induction:

Let U and W be two invariant subspaces. Then if $v \in U + W$, it can be written as $v = u + w$. Then $T(u+w) = T(u) + T(w)$. But since both U and W are invariant subspaces then $T(u) \in U$ and $T(w) \in W$. Hence $T(u+w) \in U + W$ and so $U + W$ is an invariant subspace of V .

Now, we will run an induction on the number of spaces. The base case has already been proven. Then let $W = (W_1 + \dots + W_k) + W_{k+1} = U_k + W_{k+1}$, where both of these are invariant, the first by inductive hypothesis and the second by definition. Then by our previous proof their sum is also invariant. Done.

Question 7: 5.3/6

The transition matrix with respect to the basis (recovered, dead, ambulatory, bedridden) that corresponds to this problem is:

$$M = \begin{pmatrix} 1 & 0 & 0.6 & 0.1 \\ 0 & 1 & 0 & 0.2 \\ 0 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0.2 & 0.5 \end{pmatrix}$$

The initial state is: $a = (0, 0, 0.3, 0.7)$. Then after the first month the state is:

$$Ma = (0.25, 0.14, 0.2, 0.41)$$

And so after the first month 25% of the patients recovered, 14% died, 20% were ambulatory and 41% were bedridden. To find the eventual percentages of patients we need to evaluate:

$$\lim_{n \rightarrow \infty} M^n a$$

To do this we should first diagonalize M ;

$$M = \begin{vmatrix} 1-t & 0 & 0.6 & 0.1 \\ 0 & 1-t & 0 & 0.2 \\ 0 & 0 & 0.2-t & 0.2 \\ 0 & 0 & 0.2 & 0.5-t \end{vmatrix} = (1-t)^2[(0.2-t)(0.5-t) - 0.04] = (1-t)^2(0.1-t)(0.6-t)$$

This means that the corresponding diagonal matrix is:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}$$

Now we know that $A = QDQ^{-1}$ and so $A^n = A.A \dots A = QDQ^{-1}QDQ^{-1} \dots QDQ^{-1} = QD^nQ^{-1}$. Then we can evaluate that $\lim_{n \rightarrow \infty} A^n = Q \lim_{n \rightarrow \infty} D^n Q^{-1}$ therefore:

$$L = \lim_{n \rightarrow \infty} = \begin{pmatrix} 1 & 0 & 8/9 & 5/9 \\ 0 & 1 & 1/9 & 4/9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And so we get that $59/90 \approx 65.6\%$ of patients survive and 34.4% of patients die (it is surprising they still call such a place a hospital).

Question 8: 5.3/7

The transition matrix for this problem is:

$$M = \begin{pmatrix} 1 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 0 \\ 0 & 0 & 2/3 & 1 \end{pmatrix}$$

The eigenvalues of this matrix are $1, 1, \frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}$, and the corresponding eigenvectors are:

$$a = (1, 0, 0, 0), b = (0, 0, 0, 1), c = \left(\frac{1}{2 - 3\sqrt{2}}, 1/\sqrt{2}, 1, -2(3 + \sqrt{2})/7 \right), d = \left(-\frac{1}{2 + 3\sqrt{2}}, 1/\sqrt{2}, -1, -2(-3 + \sqrt{2})/7 \right)$$

And so we can see that $e_2 = \frac{1}{\sqrt{2}} \left(c + d + \frac{3\sqrt{2}}{7}a + \frac{4\sqrt{2}}{7}b \right)$ But we know that since a, b, c, d are eigenvectors then the matrix action on them is simple: $Av = \lambda v$. And so:

$$\lim_{n \rightarrow \infty} A^n e_2 = \lim_{n \rightarrow \infty} A^n \left(c + d + \frac{3}{7}a + \frac{4}{7}b \right) = \frac{3}{7}a + \frac{4}{7}b = (3/7, 0, 0, 4/7)$$

Hence the chances of winning are $3/7$.