Solutions to PS 5 (Math 121)

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Question 1: 2.5/7

in $\mathbb{R}^2$ let L be a line $y = mx$, $m \neq 0$. Find an expression for the following $T(x, y)$'s. First let us figure out a change of basis matrix from the standard basis to the basis $a = \{(m, 1), (-1, m)\}$, because that is the basis with a vector along the line and a vector perpendicular to the line. We need to express the standard basis vectors in terms of the new basis vectors: $(1, 0) = \frac{1}{m^2+1}(m(m, 1) - (-1, m))$ and similarly $(0, 1) = \frac{1}{m^2+1}((m, 1) + m(-1, m))$ Then the change of matrix P will look like:

$$P = \frac{1}{m^2 + 1} \begin{pmatrix} m & 1 \\ -1 & m \end{pmatrix}$$

Then the inverse to P is:

$$P^{-1} = \frac{1}{m^2 + 1} \begin{pmatrix} m & -1 \\ 1 & m \end{pmatrix}$$

part a) A reflection along L

The reflection along the new basis just takes a form:

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So in the standard basis:

$$[T]_\beta = P^{-1}AP = \frac{1}{m^2 + 1} \begin{pmatrix} m^2 + m & m + 1 \\ m - m^2 & 1 - m \end{pmatrix}$$

part b) A projection onto L

Similarly in the β' basis:

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence in the β-standard basis:

$$[T]_\beta = P^{-1}AP = \frac{1}{m^2 + 1} \begin{pmatrix} m^2 & m \\ m & 1 \end{pmatrix}$$

Question 2: 2.6/13

Let us first define an **annihilator** S^0 of a subset of a vector space $S \subset V$.

$$S^0 = \{f \in V^* : \ f(x) = 0 \ \forall x \in S\}$$

1. **Prove that S^0 is a subspace of V^*.**

   (a) Clearly $0 \in S^0$.
   (b) If $a(x), b(x) \in S^0$, then this means that $a(x) = b(x) = 0$ for all $x \in S$. Therefore so will $(a+b)(x)$.
   (c) If $a(x) in S^0$ so will $ca(x)$ by the same argument.

2. **If $W \subset V$ and $x \notin W$, then show there exists $f \in W^0$ such that $f(x) \neq 0$.**
   Pick a basis for W, $\beta_W = \{v_1, \ldots, v_k\}$ and extend it onto V, $\beta_V = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$. Then form a dual basis to this basis: $(\beta_V)^* = \{f_1, \ldots, f_k, f_{k+1}, \ldots, f_n\}$. Clearly $W^0 \cap \{f_1, \ldots, f_k\}$ since $f_i(v_i) = 1 \neq 0$. On the other hand we know that $f_{k+1}, \ldots, f_n \in W^0$, since $v_{k+1}, \ldots, v_n \notin W$. But since $v \notin W$, then this means that $v = w + \sum a_i v_i$, where $i > k$ and at least one of the a_i's is non zero, let it be a_l. But then this means that $f_l(v) = a_l \neq 0$, therefore we have found one.

3. **Prove that $(S^0)^0 = \text{span}(\psi(S))$.**
   (a) **Let us first show that $\text{span}(\psi(S)) \subset (S^0)^0$.**
   Suppose that $s \in S$, then $\psi(s) = \hat{s}$. But we know that but we know that $\hat{s}(f) = f(s) = 0$. Therefore $\psi(s) \in (S^0)^0$, then by linearity of all the involved functions we know that $\sum a_i \psi(s_i) \in (S^0)^0$. Therefore $\text{span}(\psi(S)) \subset (S^0)^0$.
   (b) **It is also true that $(S^0)^0 \subset \text{span}(\psi(S))$.**
   We know that $(S^0)^0$ is a subspace of $\hat{s}$ such that $\hat{s}(f) = f(s) = 0$. Therefore it is the minimal subspace containing all $\hat{s}$, which is by definition $\text{span}(\{\hat{s}\}) = \text{span}(\psi(S))$.

4. **Show that for subspaces W_1 and W_2, then $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.**
   (a) It is clear that if $W_1 = W_2$ then $W_1^0 = W_2^0$.
   (b) Suppose that $W_1^0 = W_2^0$ and yet $W_1 \neq W_2$, that is there exists $x \notin W_1$ and $x \in W_2$. Then by part b) there must exist $f \in W_1^0$ such that $f(x) \neq 0$. But then this is contradiction since $f \notin W_2^0$.

5. **For subspace W_1 and W_2, show that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.**
   (a) Let's first prove that $(W_1 + W_2)^0 \subset W_1^0 \cap W_2^0$.
   Let $f \in (W_1 + W_2)^0$, this amongst many things means that $f(w_1) = f(w_2) = 0$, for any $w_1 \in W_1$ and $w_2 \in W_2$. Therefore $f \in W_1^0$ and $f \in W_2^0$, hence we are done.
   (b) Now, we need to prove that $W_1^0 \cap W_2^0 \subset (W_1 + W_2)^0$.
   Very similarly if $f \in W_1^0 \cap W_2^0$, then $f(W_1) = 0$ and $f(W_2) = 0$, hence $f(w_1 + w_2) = f(w_1) + f(w_2) = 0$. Done.

Question 3: 2.6/14

Show that $\dim(W) + \dim(W^0) = \dim(V)$.
There are more ways how to prove this theorem but at the end one has to, in one way or another, show a basis for W^0, so we are going to folow the hint. Besides it is a very useful construction anyway.
Pick a basis for W, $\beta_W = \{v_1, \ldots, v_k\}$ and extend it onto a basis for V, $\beta = \{v_1, \ldots v_n\}$. Then create the dual basis to β, $\beta^* = \{f_1, \ldots, f_n\}$. Then the claim is that $\alpha = \{f_{k+1}, \ldots, f_n\}$ is a basis for W^0.

1. Clearly $\alpha \subset W^0$.

2. Since α is a subset of a basis it must be linearly independent.

3. We are left to check that α spans W^0. Suppose α did not span W^0. Then since $W^0 \subset V^*$, then this would mean that some linear combination of $(\beta_W)^*$ is in W^0. This would mean that $f = \sum_{i \leq k} a_i f_i \in W^0$ and at least one of the a_i's say $a_l \neq 0$. But that is impossible since then $f(v_l) = a_l \neq 0$ although $v_l \in W$.

Hence we have the basis and we can see that $\dim(W^0) = n - k$, where $k = \dim(W)$ and $n = \dim(V)$, so:

$$\dim(V) = \dim(W^0) + \dim(W)$$

Question 4: 2.6/15

Let T be a linear transformation $T : V \to W$. Prove that $N(T^t) = (R(T))^0$.
First notice that both are subspaces of W^*. Then let us follow the standard procedure:

1. First we will prove that $N(T^t) \subset (R(T))^0$.
   If $f \in N(T^t)$, then $T^t(f) = 0$ which means that $f(T) = 0$ for $\forall v$. This means that $f(R(T)) = 0$, hence $f \in (R(T))^0$.

2. As second we will prove that $(R(T))^0 \subset N(T^t)$.
   If $f \in (R(T))^0$ then this means that $f(R(T)) = 0$, therefore this means that $f(T(v)) = 0 \; \forall v \in V$, hence this means that $T^t(f) = 0$ and so $f \in N(T^t)$.

Question 5: 2.6/16

We are trying to show that $\text{rank}(L_A) = \text{rank}(L_{A^t})$ for any $A \in M_{m\times n}(F)$.
It might be helpfull to see what this maps correspond to:

$$L_A : V \to W$$

$$L_{A^t} : W* \to V^*$$

Where $\dim(V) = \dim(V^*) = n$ and $\dim(W) = \dim(W^*) = m$. Then by our previous exrcises $\text{rank}(L_A) = \dim(R(L_A)) = \dim(W) - \dim((R(L_A))^0) = \dim(W) - \dim(N(L_{A^t})) = \dim(W) - (\dim(W^*) - \dim(R(L_{A^t})) = \dim(R(L_{A^t})) = \text{rank}(L_{A^t})$.

Question 6: 2.7/12

We shall, as usually, use the hint.

1. First, we will show that $g(D)(V) \subset N(h(D))$.
   Say that $w \in g(D)(V)$, that means there exist $v \in V$ such that $w = g(D)v$. Then suppose that $w \notin N(h(D))$. This would mean that $p(D)v = h(D)g(D)v = h(D)w \neq 0$, but this is imposible since $v \in V$.

2. We shall use lemma 2 to theorem 2.32: If T, U are two onto linear operators on $T, U : V \to V$ and their null-spaces are finite dimensional, then:

   $$\dim(N(TU)) = \dim(N(T)) + \dim(N(U))$$

   Since both our operators were proven to be onto, then we know that: $\dim(N(p(D))) = \dim(N(h(D_V))) + \dim(N(g(D_V)))$. Hence $\dim(N(p(D_V))) = \dim(V)$, and by the rank-nullity $\dim(N(g(D_V))) + \dim(R(g(D_V))) = \dim(V)$. Therefore we can see that $\dim(N(h(D_V))) = \dim(R(g(D_V)))$, which in other terms means that $\dim(N(h(D))) = \dim(R(g(D_V))) = \dim(g(D)(V))$. Since one is subspace of the other and their dimensions are equal these two subspaces must be equal.

Question 7: 3.2/5d

I will just state the progression of the augmented matrices to show both invertibility and inverse.

$$\left(\begin{array}{rrr|rrr} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|rrr} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -2 & 0 & -1 & 0 & -4 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{rrr|rrr} 2 & 0 & 2 & 1 & 2 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -2 & 0 & -1 & 0 & -4 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|rrr} -2 & 0 & 0 & 1 & 2 & -2 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -2 & 0 & -1 & 0 & -4 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -1 & 1 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -2 & 0 & -1 & 0 & -4 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -1 & 1 \\ 0 & 1 & -1 & \frac{1}{2} & 2 & -1 \\ -2 & 0 & -1 & 0 & -4 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -1 & 1 \\ 0 & 1 & -1 & \frac{1}{2} & 2 & -1 \\ 0 & 0 & -1 & -1 & -6 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -1 & 1 \\ 0 & 1 & -1 & \frac{1}{2} & 2 & -1 \\ 0 & 0 & 1 & 1 & 6 & -3 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -1 & 1 \\ 0 & 1 & 0 & \frac{3}{2} & 8 & -4 \\ 0 & 0 & 1 & 1 & 6 & -3 \end{array}\right)$$

Therefore we know this matrix is invertible and its inverse is:

$$\begin{pmatrix} -\frac{1}{2} & -1 & 1 \\ \frac{3}{2} & 8 & -4 \\ 1 & 6 & -3 \end{pmatrix}$$

Question 8: 3.2/6b

Note that $T(1) = 0$, $T(x) = (x + 1)$ and $T(x^2) = 2x(x + 1)$. Therefore the matrix for T in the standard basis is:

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Which renders it quite obvious that the rank of this matrix is 2 and so it is not invertible.

Question 9: 3.2/14

1. **Prove that $R(T + U) \subset R(T) + R(U)$ if both T, U are linear.**
   Let $w \in R(T + U)$, then this means that $w = T(v) + U(v)$ which means that $w \in R(T) + R(U)$.

2. **If W is finite dimensional, show that $\text{rank}(T + U) \leq \text{rank}(T) + \text{rank}(U)$.**
   If W is finite dimensional, then both $R(T)$ and $R(U)$ are finite dimensional as they are supbspaces of W. Therefore it makes sense to ask for the dimensional version of the statement in a). Then we can say that:

   $$\dim(R(T + U)) \leq \dim(R(T) + R(U)) \leq \dim(R(T)) + \dim(R(U))$$

   Therefore knowing that $\text{rank}(T) = \dim(R(T))$, we can see that:

   $$\text{rank}(T + U) \leq \text{rank}(T) + \text{rank}(U)$$

3. Since every matrix A has an associated linear transformation L_A and their ranks are equal, then:

   $$\text{rank}(L_{A+B}) = \text{rank}(L_A + L_B) \leq \text{rank}(L_A) + \text{rank}(L_B)$$

   And so:

   $$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$$