Problem Set 4 - Solutions

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Question 1: 2.1/26

Let $T : V \to V$ be a projection along W_1, where $V = W_1 \oplus W_2$.

part a)

T is linear. Proof:
$T(x) = T(x_1+x_2) = x_1$, where $x_1 \in W_1$. Then $T(a+b) = T(a_1+b_1+a_2+b_2) = a_1+b_1 = T(a)+T(b)$. The first stap is possible because each a can be split into $a_1 \in W_1$ and $a_2 \in W_2$, and clearly then if $a+b = w_1+w_2$, then $w_1 = a_1+b_1 \in W_1$ and $w_2 = a_2+b_2 \in W_2$. Claim: $W_1 = \{x \in V : T(x) = x\}$. Proof:
If $w \in W_1$, then $T(w) = w$ by definition. Hence $W_1 \subset \{x \in V : T(x) = x\}$. On the other hand, suppose that $T(x) = x$, but $x \notin W_1$. Then this means $T(x) = T(x_1 + x_2) = x_1$, which implies that $x_2 = 0$. But this means that $x = x_1 \in W_1$ - contradiction. Therefore: $W_1 = \{x \in V : T(x) = x\}$.

part b)

By the definition of the map $R(T) \subset W_1$. However, since $T(W_1) = W_1$, then $W_1 \subset R(T)$. Together this means $R(T) = W_1$.
Suppose $v \in W_2$, then $T(w) = 0$, hence $W_2 \subset N(T)$. Yet at the same time, if $T(x) = 0$, then $T(x_1 + x_2) = x_1 = 0$, which means $x_1 = 0$, $x = x_2 \in W_2$. We can conclude $N(T) = W_2$.

part c)

If $W_1 = V$, then $W_2 = 0$ and also $T(x) = x$ for any x, hence $T = I$.

part d)

If $W_1 = 0$, then $N(T) = W_2 = V$, therefore $T(x) = 0$ for all x.

Question 2: 2.2/12

Let T still be a projection on W along W', where $V = W \oplus W'$. Then we can pick a basis for the subspace W, $\beta_W = \{w_1, \ldots, w_k\}$ and extend it onto a basis for V, $\beta_V = \{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$. Then we know that the basis for W' is the set $\{w_{k+1}, \dots, w_n\}$ (it is certainly linearly independent as it is a part of another basis, it also must span since $V = W \oplus W'$). Then, by the previous exercise we know that: $T(w_i) = w_i$ and $T(w_j) = 0$ for all i or j. Therefore the matrix with repsect to this basis will look like:

$$[T]_\beta = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

Where the rest of entries is zeros.

Question 3: 2.3/12

Let V, W, Z be vectorspaces, $T : V \to W$ and $U : W \to Z$.

part a)

Show that if UT is one-to-one then T is one-to-one.
Proof: Suppose T is not one to one and UT is one to one. Then there exist $v_1 \neq v_2 \in V$ such that $T(v_1) = T(v_2)$. Then $UT(v_1) - UT(v_2) = UT(v_1 - v_2) = U(T(v_1) - T(v_2)) = U(0) = 0$ which implies $UT(v_1) = UT(v_2)$ and so UT is not one to one - a contradiction.
However, U does not have to be one-to-one. An example is: $T : P_2 \to P_3$ as an inclusion which is trivially linear and $U : P_3 \to P_2$, by truncation of the x^3 term, which is again linear, but also not one-to-one as $N(U) = \text{span}(x^3) \neq 0$. But $UT = I$ which is trivially one-to-one.

part b)

Suppose that U is not onto but UT is onto. This is contradiction of the start, since if U does not hit all the vectors in Z nor can UT.
However, T does not have to be onto. Consider $T : P_2 \to P_3$ as inclusion and $U : P_3 \to P_1$ a projection. Then T is not onto since $\dim(R(T)) = 3$, whereas $\dim(P_3) = 4$. But UT is clearly onto.

part c)

Assume both U, T are both onto and one to one.
Claim: UT is onto.
Proof: Pick $z \in Z$, then since U is onto then there exists $w\ inW$, s.t $U(w) = z$. But the there is also $x \in X$ s.t. $T(x) = w$. Hence $UT(x) = U(w) = z$. Therefore UT is onto.

Claim: UT is one to one.
Proof: Since $N(U) = \{0\}$, then $N(UT) = N(T)$. But since $N(T) = \{0\}$ as T is one to one, then $N(UT) = \{0\}$. Therefore UT is one-to-one.

Question 4: 2.3/13

The trace of a matrix is defined as:

$$\mathrm{Tr}(A) = \sum A_{ii}$$

Claim: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.
Proof: By definition of matrix multiplication:

$$(AB)_{ij} = \sum A_{ik} B_{kj}$$

Hence

$$\mathrm{Tr}(AB) = \sum \sum A_{ik} B_{ki}$$

On the other hand

$$\mathrm{Tr}(BA) = \sum \sum B_{ik} A_{ki}$$

But since C_{ij} are all scalars, they commute so the equations are the same.
Claim: $\mathrm{Tr}(A^T) = \mathrm{Tr}(A)$.
Proof: Since the diagonal is unchanged by transposition and the trace is a sum of elements along the diagonal, nothing changes by transposing.

Question 5: 2.3/21

Unfortunately there is no nice enough linear algebra proof. However, there is an elegant and short graph proof. We shall prove the following statement that is equivalent to our question.
Statement: The person who dominates most people directly dominates all in the first and second stage.
Proof: Assume to the contrary: There exists at least one person Z who is not dominated by a person A who dominates most directly. This means that Z must dominate A directly, since otherwise A would dominate Z. Also this means that whoever is dominated by A must be also dominated by Z, since otherwise Z would be dominated by A in the second stage. But this means that Z directly dominates all who A does directly plus it directly dominates A, hence Z dominates at least one more person directly than A. That's a contradiction, since A was supposed to directly dominate the most people.

Question 6: 2.4/3

part a)

F^3 cannot be isomorphic to $P_3(F)$, because $\dim(F^3) = 3$, $\dim(P_3(F)) = 4$. Hence they cannot be isomorphic.

part b)

F^4 and $P_3(F)$ are isomorphic, because $\dim(F^4) = 4$, $\dim(P_3(F)) = 4$. Therefore there exists an isomorphism by our theorem from class.

part c)

$M_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$ are isomorphic, because $\dim(M_{2\times 2}(\mathbb{R})) = 4$, $\dim(P_3(\mathbb{R})) = 4$. Therefore there exists an isomorphism by our theorem from class.

part d)

We can show that $\dim(V) = 3$, because our space is made out of matrices of form:

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

Hence the basis consists of a subset of the following matrices, since they clearly span the space:

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

Hence $\dim(V) = 3 \neq 4 = \dim(\mathbb{R}^4)$.

Question 7: 2.4/5

It would be very helpfull to prove the previous exrcise first:

SubQuestion 7.1: 2.4/4

Claim: Let A, B be invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
Proof: We need to check the inverses work: $AB(AB)^{-1} = AB(B^{-1}A^{(}-1)) = ABB^{-1}A^{-1}$ But since $BB^{-1} = I$, then $ABB^{-1}A^{-1} = AA^{-1} = I$. Similarly $(AB)^{-1}AB = B^{-1}A^{-1}AB = B^{-1}B = I$.

SubQuestion 7.2: 2.4/5

Now, we can proceed with proving the main result.
Claim: If A is invertible the inverse of A^T is $(A^{-1})^T$. Hence A^T is invertible.
Proof: We know that since A is invertible there exists A^{-1} such that $AA^{-1} = I$ and $A^{-1}A = I$. We can transpose both equations to get: $(A^{-1})^T A^T = I^T$ and $A^T(A^{-1})^T = I^T$, since $(AB)^T = B^T A^T$. But $I^T = I$ and so we can read that: $(A^{-1})^T A^T = I$ and $A^T(A^{-1})^T = I$, which means that $(A^{-1})^T$ is an inverse to A^T, in our notation then $(A^T)^{-1} = (A^{-1})^T$, and so A^T is invertible.

Question 8: 2.4/17

Let V, W be finite dimensional spaces and $T : V \to W$ is an isomorphism. Let Z be a subspace of V.

part a) Prove that $T(Z)$ is a subspace of W.

First we can check that $T(Z)$ contains 0_W since $0_V \in Z$ and $T(0_V) = 0_W$. Then we need to check the closure under addition. Suppose, $w_1, w_2 \in T(Z)$, then there exist $T(z_1) = w_1$ and $T(z_2) = w_2$, such that $z_1, z_2 \in Z$. But this means that $z_1 + z_2 \in Z$ and hence $T(z_1+z_2) = T(z_1)+T(z_2) = w_1+w_2 \in T(Z)$ hence $T(Z)$ is closed under addition and similarly under scalar multiplication. Knowing this we can conlcude $T(Z)$ is a subspace of W. Note that we did not need to use any part of the isomorphism condition yet, so this means that every linear map sends subspaces onto subspaces.

part b) Show that $\dim(Z) = \dim(T(Z))$.

Proof: There are many ways to prove this theorem that involve showing that the basis of Z stays still linearly independent, since T is an isomorphism and then comparing the number of elements of this basis. However if we define the term restriction of T then we can use a theorem from class about equidimensionality of isomorphic spaces.
Define $T_Z : Z \to T(Z)$, to be a restriction of the map T to a map defined the same way but only on $Z \subset V$. Then by definition T_Z is onto and since $N(T) = \{0\}$, then $N(T_Z) = \{0\}$. Hence T_Z is an isomorphism since it is invertible. Moreover it maps Z onto $T(Z)$, hence these two are isomorphic and therefore must have the same dimension.

Question 9: 2.4/22

Using the hint really helps with this problem. The definition of Lagrange polynomials is:

$$f_i = \prod_{k\neq i}^{0\leq k\leq n} \frac{x - c_k}{c_i - c_k}$$

Therefore $f_i(c_j) = \delta_{ij}$ also it turns out that for any set of distinct of $\{c_0, \ldots, c_n\}$ the set of associated Lagrange polynomials is a basis for $P_n(F)$ (pages 51-53). Hence by the theorem in class $R(T)$ is generated by $\{T(f_0), \ldots, T(f_n)\}$, but note that $T(f_i) = e_{i+1}$ thanks to the property $f_i(c_j) = \delta_{ij}$. Hence $\{T(f_0), \ldots, T(f_n)\} = \{e_1, \ldots, e_{n+1}\}$, but since this set is linearly independent, then $\dim(R(T)) = n + 1 = \dim(P_n(F)) = \dim(F^{n+1})$. Hence T is onto and by the rank-nullity theorem it is one-to-one. Therefore T is an isomorphism.