Problem Set 4.ϵ - Solutions

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Question 1: 2.5/8

We can pretty much follow the proof of theorem 2.23:
Let $T : V \to W$ and β, β' be the ordered bases for V and similarly with γ's for W. Then we can consider the transformation: $[T]^{\gamma}_{\beta}Q = [T]^{\gamma}_{\beta}[I]^{\beta}_{\beta'} = [T]^{\gamma}_{\beta'}$ We can also check that P^{-1} switches γ to γ': $PP^{-1} = I_{\gamma}$, hence the result is in γ basis but the argument of P is in the γ' basis. Moreover: $PP^{-1} = [I]^{\gamma}_{\gamma'}P^{-1} = I_{\gamma}$, therefore $P^{-1} = [I]^{\gamma'}_{\gamma}$. And thence by repeating the argument from first part:

$$[T]^{\gamma'}_{\beta'} = P^{-1}[T]^{\gamma}_{\beta}Q$$

Question 2: 2.5/10

If A and B are similar matrices, then there exists an invertible matrix Q, such that $B = Q^{-1}AQ$. Then by exercise 13 in section 2.3: $\mathrm{Tr}(CD) = \mathrm{Tr}(DC)$, hence:

$$\mathrm{Tr}(B) = \mathrm{Tr}(Q^{-1}AQ) = \mathrm{Tr}(AQQ^{-1}) = \mathrm{Tr}(AI) = \mathrm{Tr}(A)$$

Question 3: 2.6/4

Let $V = \mathbb{R}^3$, and let there be $f_1, f_2, f_3 \in V^*$ such that:

$$\begin{aligned} f_1 &= x - 2y \\ f_2 &= x + y + z \\ f_3 &= y - 3z \end{aligned}$$

Since we know that $\dim(V^*) = \dim(V) = 3$, since V is finite dimensional, then it is enough to show span or linear independence. Let us show linear independence. Then let us explore:

$$a(x - 2y) + b(x + y + z) + c(y - 3z) = (a + b)x + (b + c - 2a)y + (b - 3c)z = 0$$

Since this is true for all (x, y, z), the chose e_i which gives us $a = -b$, $3b + c = 0$ and $b - 3c = 0$, hence $a = b = c = 0$ is the only solution. Therefore f_1, f_2, f_3 form a basis for V^*.

part b) Dual basis to the basis of the dual :-)

We need to find vectors in $e_1, e_2, e_3 \in V$ that have the following properties:

$$\begin{array}{lll} f_1 e_1 = 1 & f_2 e_1 = 0 & f_3 e_1 = 0 \\ f_1 e_2 = 0 & f_2 e_2 = 1 & f_3 e_2 = 0 \\ f_1 e_3 = 0 & f_2 e_3 = 0 & f_3 e_3 = 1 \end{array}$$

We will find the first one. We know that $x_1 - 2y_1 = 1$, $x_1 + y_1 + z_1 = 0$ and $y_1 - 3z_1 = 0$, then the last tow give us: $4y_1 = -3x_1$, hence plugged into the first equation we get: $e_1 = (2/5, -3/10, -1/10)$. Then the other two can be found in a precisely the same way: $e_2 = (6/10, 3/10, 1/10)$ and $e_3 = (2/10, 1/10, -3/10)$.

Question 4: 2.6/12

Let β be an ordered basis for V and $\psi : V \to V^{**}$ such that $\psi(x) = \hat{x}$. Then we are to show that $\psi(\beta) = \beta^{**}$. So we are trying to show that given β, then β^* and β^{**} are the associated dual bases. In other words if $v_i \in \beta$, and $f_j \in \beta^*$ such that $f_j(v_i) = \delta_{ij}$, then $\psi(v_i) f_j = \hat{v_i} f_j = \delta_{ij}$. But this is true since $\hat{v_i} f_j = f_j(v_i) = \delta_{ij}$.

Question 5: 2.7/13

part a) Onto?

By lemma 1 to theorem 2.32, we know that the operator $(D - cI)$ is onto for any complex number c. Also by the fundamental theorem of algebra we know that every polynomial $p(t) \in C^\infty$ can be factored into linear factors. Thereofore $p(t) = \prod(t - c_i)$ and so $p(D) = \prod(D - cI)$. Since each of these operators is onto, then the resulting operator $p(D)$ (by induction on the fact that composition of onto functions is an onto function from last problem set). Hence $p(D)$ is onto, which means that for any $x \in C^\infty$, there exists y such that $p(D)y = x$.

part b) Solution set

By definition $p(D)y = 0$, and $p(D)z = x$. Then $p(D)(y + z) = p(D)y + p(D)z = 0 + x = x$, therefore $z + y$ is a solution. Now we need to check that these are the only solutions. Consider $p(D)w = x$. Then $p(D)(w - z) = p(D)w - p(D)z = x - x = 0$, therefore $p(D)(w - z) = 0$, which means $w - z \in N(p(D))$, but $N(p(D)) = V$ by definition.

Question 6: 2.7/18

We are trying to solve the differential equation $my'' + ry' + ky = 0$, where $r > 0$. Then the associated polynomial is: $mx^2 + rx + k = 0$, whose solutions are:

$$c_{1,2} = \frac{-r \pm \sqrt{r^2 - 4mk}}{2m}$$

Hence we encounter two cases: if $r^2 = 4mk$ then the solution space has basis $\{e^{-\frac{rt}{2m}}, te^{-\frac{rt}{2m}}\}$, otherwise the basis for the solution space is $\{e^{c_1 t}, e^{c_2 t}\}$. Therefore the two cases will give us two mutually exclusive general solutions:

$$\begin{aligned} y_{repeated} = y_r &= e^{-\frac{rt}{2m}}\left[A + Bt\right] \\ y_{normal} = y_n &= e^{-\frac{rt}{2m}}\left[Ce^{\frac{\sqrt{r^2-4mk}}{2m}t} + De^{-\frac{\sqrt{r^2-4mk}}{2m}t}\right] \end{aligned}$$

part b) and c) Find a specific solution such that $y(0) = 0$ and $y'(0) = v_0$ and show that it dies away.

Let us explore the first case:

$$y_r(0) = A = 0$$

Hence $A = 0$. Then $y_r'(0) = v_0$ yeilds:

$$v_0 = y_r'(0) = B$$

Therefore, if there is a double root then:

$$y_r(t) = v_0 te^{-\frac{rt}{2m}}$$

And clearly

$$\lim_{t\to\infty} y_r(t) = 0$$

What if there is no repeated root? In that case we need to consider:

$$y_n = e^{-\frac{rt}{2m}}\left[Ce^{\frac{\sqrt{r^2-4mk}}{2m}t} + De^{-\frac{\sqrt{r^2-4mk}}{2m}t}\right]$$

But $y_n(0) = C + D = 0$ hence we know that $C = -D$. Differentiating we'll get:

$$y_n'(0) = C\frac{\sqrt{r^2-4mk}}{m} = v_0$$

Therefore: $C = \frac{v_0 m}{\sqrt{r^2-4mk}}$ and so the solution goes:

$$y_n(t) = \frac{v_0 me^{-\frac{rt}{2m}}}{\sqrt{r^2-4mk}}\left[e^{\frac{\sqrt{r^2-4mk}}{2m}t} - e^{-\frac{\sqrt{r^2-4mk}}{2m}t}\right]$$

For evaluating the limit, first observe that $r^2 > r^2 - 4mk$ and so the whole expression is dominated by the behaviour of the first exponential hence

$$\lim_{t\to\infty} y_n(t) = 0$$