Problem Set 3 - Solutions

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Question 1: 1.3/31

a) Cosets and Subspaces

We want to show that $v + W$ is a subspace if and only if $v \in W$.
($\Leftarrow$) Suppose that $v+W$ is a subspace. $v+W$ must contain 0. Then there exists $u \in W$ such that $v + u = 0$, hence W contains $-v$, and sincd it is a subspace itself then W contains also v.
($\Rightarrow$) If $v \in W$, then the set of form $\{v + w, w \in W\} = W$, since that is closed under addition. Therefore $v + W = W$ which we know is a subspace.

b) Equivalent representations

Suppose that $v_1 + W = v_2 + W$, then this is equivalent to the statement that for every $w_1 \in W$, there exists $w_2 \in W$ such that $v_1 + w_1 = v_2 + w_2$, which is equivalent with the statement $v_1 - v_2 = w_2 - w_1$, but this is equivalent with the statement that $v_1 - v_2 \in W$ since $w_2 - w_1 \in W$ since W is a suspace. Since all these statements were equivalent then the if and only condition is satisfied.

c) Are these operations well defined?

If $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$ then $v_1 - v_1' \in W$ and $v_2 - v_2' \in W$. Therefore $(v_1 + v_2) - (v_1' + v_2') \in W$, hence $(v_1 + v_2) + W = (v_1' + v_2') + W$, which in turn means that $(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$.
For scalar multiplication: if $v_1+W = v_1'+W$ then $v_1-v_1' \in W$, hence $a(v_1-v_1') \in W$ therefore $av_1 - av_1' \in W$ which in turn implies that $av_1 + W = av_1' + W$. But this is equivalent to the statement: $a(v_1 + W) = a(v_1' + W)$.

d) They form a vector space!

We need to check all the axioms:
VS1(Commutativity): $(v + W) + (u + W) = (v + u) + W = (u + v) + W = (u + W) + (v + W)$, where we used the fact that V is a vectorspace.
VS2(Associativity): Use the same trick knowing that V is a vectorspace.
VS3(Existence of 0): The coset $0 + W$ does the job since $(0 + W) + (a + W) = (a + 0) + W = a + W$.

VS4(Inverses): Use the inverses from V: $(-v+W)+(v+W)=(v-v)+W=0+W$.
VS5(Identity): Take the identity from V: $1(v+W)=1v+W=v+W$.
The last three are inherited due to the nature of definition of addition and scalar multiplication in V/W.

Question 2: 1/6/33

a)

Let $V=W_1\oplus W_2$, that means that $W_1\cap W_2=0$. Now suppose that $v\in\beta_1\cap\beta_2$, this would mean that $v\in W_2$ and $v\in W_1$, which is a contradiction.
Now we know that $V=W_1\oplus W_2$, hence if $v\in V$, then there exist vectors $a\in W_1$ and $b\in W_2$ such that $a+b=v$. But since $a\in \text{span}(\beta_1)$ and $b\in\text{span}(\beta_2)$, then $a+b\in \text{spann}(\beta_1\cup\beta_2)$, hence $\beta_1\cup\beta_2$ spans V. We need to show that $\beta=\beta_1\cup\beta_2$ is linearly independent. It was not, then there would be a linear combiantion of vectors in β that is equal to the zero vector. Then this means that $0=\sum a_iv_i+\sum b_ju_j=v+u$, where $v_i\in\beta_1$ and $u_j\in\beta_2$. But this would mean that for any vector $c\in W_1\oplus W_2$ such that $c=a+b$, $a\in W_1$ and $b\in W_2$, it holds true that $c=c+0=a+b+(0)=a+b+u+v=(a+u)+(b+v)$, which is a contradiction since this expression is supposed to be unique.

b)

Suppose that β_1 and β_2 are disjoint sets and $\beta=\beta_1\cup\beta_2$ is a basis for V. Since the basis are disjoint and their union forms a basis for V, then $W_1\cap W_2=0$. Then if $v\in V$, therefore $v\in\text{span}(\beta)$ hence $v=a+b$, where $a\in\text{span}(\beta_1)$ and $b\in\text{span}(\beta_2)$. Therefore any $v\in V$ can be expressed as a sum of vectors from W_1 and W_2, therefore $V\subset W_1\oplus W_2$. On the other hand if $v\in W_1\oplus W_2$, then $v=a+b$, $a\in W_1$ and $b\in W_2$ and so $v\in\text{span}(\beta_1\cup\beta_2)$. Therefore $V=W_1\oplus W_2$.

Question 3: 1.6/35

Let $W\subset V$ as a subspace and the basis of W be $\beta_W=\{u_1,\dots,u_k\}$. Then let the extension of this basis onto a basis of V be $\beta_V=\{u_1,\dots,u_k,u_{k+1},\dots,v_n\}$. The show that the following set is a basis for V/W:

$$\beta_{V/W}=\{u_{k+1}+W,\dots,u_n+W\}$$

Claim: $\beta_{V/W}$ spans V/W.
Proof: The span of this basis is:

$$\sum a_{k+i}(u_{k+i}+W)=(\sum a_{k+i}u_{k+i})+W$$

But these are precisely all the cosets of W since we know that $v_1+W=v_2+W\ \Leftrightarrow\ v_1-v_2\in W$. So $(v_1)+W=(v_1+w)+W$, where $w\in W$. Hence the span of our basis is in fact:

$$\sum a_{k+i}(u_{k+i}+W)=\left(\sum a_{k+i}u_{k+i}\right)+W=\left(\sum a_{k+i}u_{k+i}+w\right)+W$$

$$= \left(\sum a_{k+i} u_{k+i} + \sum_{j \leq k} a_j u_j \right) + W = \left(\sum a_i u_i \right) + W$$

Which is a general vector of form $v + W$.
Claim: $\beta_{V/W}$ is linearly independent.
Proof: We want to find the solutions of the the linear combination of the set equal to the zero coset:

$$\sum a_{k+i}(u_{k+i} + W) = \left(\sum a_{k+i} u_{k+i} \right) + W = 0 + W$$

Which means:

$$\sum a_{k+i} u_{k+i} \in W$$

(Note: this is important because it does not necessarily mean $\sum a_{k+i} u_{k+i} = 0$)
But then we know that none of the u_{k+i}'s is in W since if they were than they would be in the span of β_W, which would mean that β_V could not be linearly independent. Hence $\text{span}(u_{k+i}, \ldots, u_n) \cap W = \{0\}$. Therefore

$$\sum a_{k+i} u_{k+i} = 0$$

Hence all the a_{k+i}'s must be zero since u_i's are a part of the basis and thence linearly independent. This establishes the fact that $\beta_{V/W}$ is really a basis of V/W.

part b)

By counting the number of elements of each basis we can reach the conclusion that:

$$\dim(V/W) = \dim V - \dim W$$

Question 4: 2.1/5

Let $T : P_2 \to P_3$ be defined as:

$$T(f(x)) = xf(x) + f'(x)$$

T is linear since:

$$T(af(x) + bg(x)) = x(af(x) + bg(x)) + (af(x) + bg(x))' =$$

$$= a(xf(x) + f'(x)) + b(xg(x) + g'(x)) = aT(f) + bT(g)$$

The solution to this differential eq is of the form $h(x) = Ae^{-x^2/2}$ but $h(x) \notin P_2$ unless $A = 0$, therefore $N(T) = 0$. On the other hand, we know that the basis of the range is a subset of the image of the original basis: $\{T(1), T(x), T(x^2)\} = \{x, x^2+1, x^3+2x\}$. This set is linearly independent by the exchange lemma, and therefore it is a basis of the range. Therefore $\text{rank}(T) = 3$ and $\text{nullity}(T) = 0$. This satisfies the rank-nullity theorem since $3 = 3 + 0$. Also by the theorem 2.4 T is one-to-one. On the other hand since $\dim R(T) < \dim P_3$, then it cannot be onto.

Question 5: 2.1/11

Since $v_1 = (1,1)$ and $v_2 = (2,3)$ are linearly independent (one is not multiple of the other), they form a basis $\alpha = \{v_1, v_2\}$ of $\mathbb{R}^2$, which means that we can construct a linear transformation such that $T(v_1) = (1,0,2)$ and $T(v_2) = (1,-1,4)$, by theorem 2.6. Note that $(8,11) = 2(1,1) + 3(2,3)$ hence $T(8,11) = 2(1,0,2) + 3(1,-1,4) = (5,-3,16)$.

Question 6: 2.1/12

No there is no such linear transformation. Assume there was one: then $(1,1) = T((-2,0,-6)) = -2T((1,0,3)) = (-2,-2)$, but then $T((-2,0,-6)) = (2,1)$ - a contradiction.

Question 7: 2.1/18

Give an example of a linear transformation such that $N(T) = R(T)$.
Example: Consider the transformation $T(a,b) = (a-b, a-b)$.
First, T is linear since $T(a+c, b+d) = (a+c-(b+d), a+c-(b+d)) = (a-b, a-b) + (c-d, c-d)$, similarly for scalar multiplication. Secondly the range is clearly of form (c,c), hence $R(T) = \text{span}((1,1))$. On the other hand $T(a,b) = (0,0)$ implies that $a = b$, therefore $N(T) = \text{span}((1,1))$.

Question 8: 2.1/21

Let V be the (infinite dimensional) space of all the sequences with only finitely many nonezero entries. Define the two operators:

$$T(a_1, a_2 \ldots) = (a_2, a_3, \ldots)$$

$$U(a_1, a_2 \ldots) = (0, a_1, a_2, \ldots)$$

part a) Show that both T and U are linear:

Lets examine two sequences a_i and b_i, then:

$$U(a_i + b_i) = U(a_1 + b_1, a_2 + b_2, \ldots) = (0, a_1 + b_1, a_2 + b_2, \ldots) =$$

$$= (0, a_1, a_2, \ldots) + (0, b_1, b_2, \ldots) = T(a_i) + T(b_i)$$

For scalar multiplication:

$$U(ca_i) = U(ca_1, ca_2, \ldots) = (0, ca_1, ca_2, \ldots) = c(0, a_1, a_2, \ldots) = cU(a_i)$$

Similarly for T.

part b) Show that T is onto but not one-to-one

T is not one-to-one simply because we knwo that $(1, 0, 0 \ldots) \in N(T)$, and so by theorem 2.4 it cannot be one-to-one. On the other hand if we have a sequence a_i, than it certainly is in the range since the sequence $b_i = a_{i-1}$, and $b_1 = 0$, gets mapped onto a_i.

part c) Show that U is not onto but is one-to-one

Note that the range of U contains only vectors of form $(0, a_1, a_2, \ldots)$. Hence $R(T) \neq V$ and so it is not onto. On the other hand the $U(a_i) = 0$ iff $a_i = 0$ since otherwise there will be some nonzero a_i and hence so $U(a_i) = (0, a_i) \neq 0$. Hence $N(T) = 0$ and then by theorem 2.4 it is one-to-one.

Question 9: 2.1/35

Let V be finite dimensional and T be linear.

part a)

Assume $V = R(T) + N(T)$. By the fact from section

$$\dim(W + U) = \dim(W) + \dim(U) = \dim(W \cap U)$$

Hence in terms of $N(T), R(T)$ and by rank nullity theorem both of these hold:

$$\begin{aligned} \dim(N + R) = \dim(V) &= \dim(R) + \dim(N) - \dim(N \cap R) \\ \dim(V) &= \dim(R) + \dim(N) \end{aligned}$$

Comparing these two we get:

$$\dim(R \cap N) = 0$$

Hence $V = R + N$ and $R \cap N = 0$, therefore $V = R \oplus N$.

part b)

Similarly assume $R \cap N = 0$. Then by the fact from section and the rank nullity theorem:

$$\begin{aligned} \dim(N + R) &= \dim(R) + \dim(N) \\ \dim(V) &= \dim(R) + \dim(N) \end{aligned}$$

Hence $\dim(N + R) = \dim(V)$. Since both $N \subset V$ and $R \subset V$, then $N + R \subset V$. But since their dimensions are the same then $N + R = V$. Moreover, since by assumption $N \cap R = 0$, then $V = N \oplus R$.

Question 10: 2.2/3

Let T be a map such that $T(a,b) = (a-b, a, 2a+b)$. Let β be the standard basis of $mathbbR^2$. Also let $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$. Also let β_3 be the standard basis for $\mathbb{R}^3$.

part i) Compute $[T]_\beta^\gamma$

Start with $T(e_1) = (1,1,2)_{\beta_3} = -\frac{1}{3}(1,1,0)+\frac{2}{3}(2,2,3)$, then $T(e_2) = (-1,0,1)_{\beta_3} = -(1,1,0)+(0,1,1)$. Therefore the matrix will look like:

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$$

part ii) Compute $[T]_\alpha^\gamma$

$T((1,2)) = (-1,1,4) = \frac{1}{3}(-7(1,1,0)+6(0,1,1)+2(2,2,3))$, then similarly $T((2,3)) = (-1,2,7) = \frac{1}{3}(11(1,1,0)+9(0,1,1)+4(2,2,3))$. Therefore:

$$[T]_\beta^\gamma = \frac{1}{3}\begin{pmatrix} -7 & -11 \\ 6 & 9 \\ 2 & 4 \end{pmatrix}$$

Question 11: 2.2/10

If $\beta = \{v_1, \ldots, v_n\}$ and T is linear and such that $T(v_i) = v_i + v_{i-1}$ and $v_0 = 0$. Then by the very way this is written, $T(v_1) = v_1$, $T(v_2) = v_1 + v_2$ and so on..

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & & 0 \\ 0 & 0 & 1 & \ddots & \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$