Problem Set 2 - Solutions

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Question 1: 1.5/4

In order to show that the set $\{e_1, \ldots, e_n\}$ is linearly independent we need to show that the following equation is satisfied only if $c_1 = \ldots = c_n = 0$:

$$c_1 e_1 + c_2 e_2 + \ldots + c_n e_n = 0$$

But since $e_1 = (1, 0, \ldots, 0)$ and so on, then the LHS is equal to $(c_1, c_2, \ldots, c_n)$. Hence the last equation reads: $(c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0)$ Which can only be satisfied if all c_is are zero.

Question 2: 1.5/13

a) Two vectors

Claim: If u and v are linearly independent then so are $u + v$ and $u - v$.
Proof: The equation $a(u+v)+b(u-v) = 0$ expands as $(a+b)u+(a-b)v = 0$, but since u, v are linearly independent by our assumption, then $a + b = 0$ and $a - b = 0$, as F is not of charcteristic 2 then $a = -b$ and $a = b$ imply $a = b = 0$ and so our vectors are linearly independent.

Claim: If $u + v$ and $u - v$ are linearly independent then so are u and u.
Proof: If $u + v$ and $u - v$ are linearly independent then so are $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. (Dividing by 2 is posssible since $\mathrm{Char}(F) \neq 2$). Then observe that $u = x + y$ and $v = x - y$ and use the previous part.

b) Three vectors

This is almost the same as the previous part. Please see me if you need more explanation.

Question 3: 1.5/17

An upper triangular matrix is of form:

$$A = \begin{pmatrix} c_{11} & c_{21} & \dots & c_{n1} \\ 0 & \ddots & \dots & c_{n2} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & c_{nn} \end{pmatrix}$$

Then the collumns form vectors of form $v_i = (0, \dots, 0, k_{ii}, \dots, k_{in})$ where the first $i-1$ elements are zeroes. This means that their linear combination gives:

$$a_1 v_1 + \ldots + a_n v_n = (a_1 k_{11}, a_1 k_{12} + a_2 k_{22}, \dots, \sum a_j k_{jn}) = 0$$

But $k_{11} \neq 0$ since the diagonal entries are non-zero, hence $a_1 = 0$. Then the second entry reads $a_2 k_{22} = 0$, and agains since $k_{22} \neq 0$,then $a_2 = 0$. This process can be repeated finitely many times (n-times) to show that $a_1 = \ldots = a_n = 0$ and hence the set is linearly independent.

Question 4: 1.6/3(d-e)

part d)

We can check, the only solution to:

$$a(-1 + 2x + 4x^2) + b(3 - 4x - 10x^2) + c(-2 - 5x - 6x^2) = 0$$

is $a = b = c = 0$. Hence these three polynomials are linearly independent. And since we know that $\dim(P_2(\mathbb{R})) = 3$. Then we know these three polynomials form a basis.

part e)

We can see that:

$$-7(1 + 2x - x^2) + 2(4 - 2x + x^2) + (-1 + 18x - 9x^2) = 0$$

Therefore this set is not linearly independent and so it cannot be a basis.

Question 5: 1.6/14

a) Case 1:

Let $W_1 = \{(a_1, \dots, a_5) \in F^5, a_1 - a_3 - a_4 = 0\}$. The my claim is that the set $\{(1,0,0,1,0), (0,1,0,0,0), (0,0,1,-1,0), (0,0,0,0,1)\}$ is a basis. Checking span:

$$a_1(1,0,0,1,0) + a_2(0,1,0,0,0) + a_3(0,0,1,-1,0) + a_5(0,0,0,0,1) = (a_1, a_2, a_3, a_1 - a_3, a_5)$$

but since $a_4 = a_1 - a_3$ this is a general vector in W_1. Checking Linear independence:

$$a_1(1,0,0,1,0)+a_2(0,1,0,0,0)+a_3(0,0,1,-1,0)+a_5(0,0,0,0,1) = (a_1, a_2, a_3, a_1-a_3, a_5)$$

so if this is to be equal to zero then $a_1 = a_2 = a_3 = a_5 = 0$ - the set is linearly independent. Therefore W_1 has dimension 4.

b) Case 2:

Let $W_2 = \{(a_1, \ldots, a_5) \in F^5, a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}$. The my claim is that the set $\{(0,1,1,1,0),(1,0,0,0,-1)\}$ is a basis. One needs to check that this set spans W_2 and is linearly independent: but the procedure is the same as in a) so I'll skip it (you should not in your problem set). Hence W_2 is two-dimensional.

Note: *Observe that a 5-dimensional space in both cases gets reducet to a 4 and 2-dimensional subspace by imposing conditions. In the first case there was only 1 condition and hence $4 = 5 - 1$ and in the second case there were three conditions and $2 = 5 - 3$, such a reasoning can help you to build your intuition about the dimensionality of resultinh spaces but is unfortunately not rigorous enough to be used as a proof.*

Question 6: 1.6/17

A general skew symmetric matrix satisfies the equation $A^T = -A$, which implies that $a_{ij} = -a_{ji}$. Therefore it takes a form of:

$$A = \begin{pmatrix} 0 & c_{21} & \cdots & c_{n1} \\ -c_{21} & \ddots & c_{32} & c_{n2} \\ \vdots & -c_{32} & \ddots & \vdots \\ -c_{n1} & \vdots & -c_{n,n-1} & 0 \end{pmatrix}$$

Hence it seems the basis should be all the matrices such that A_{ij} is a mtrix with $a_{ij} = -a_{ji} = 1, i \neq j$ and all other entries equal to zero. By counting the number of entries we can see that there are n^2 entries, then take away n for the diagonal entries and divide by 2 due to the symmetry. Hence there are $N = \frac{n(n-1)}{2}$ of these matrices. However, we need to prove they really form a basis. Let's start with span:

$$\sum_{i<j} c_{ij} A_{ij} = \begin{pmatrix} 0 & c_{21} & \cdots & c_{n1} \\ -c_{21} & \ddots & c_{32} & c_{n2} \\ \vdots & -c_{32} & \ddots & \vdots \\ -c_{n1} & \vdots & -c_{n,n-1} & 0 \end{pmatrix}$$

Hence they span. Checkign linear independence:

$$\sum_{i<j} c_{ij} A_{ij} = \begin{pmatrix} 0 & c_{21} & \dots & c_{n1} \\ -c_{21} & \ddots & c_{32} & c_{n2} \\ \vdots & -c_{32} & \ddots & \vdots \\ -c_{n1} & \vdots & -c_{n,n-1} & 0 \end{pmatrix}$$

if this is to be equal to the zero matrix then clearly $c_{ij} = 0$ for all i, j. Hence this is a bases and the skew symmetric $n \times n$ matrices form a subspace of dimension $\frac{n(n-1)}{2}$.

Question 7: 1.6/22

It turns out that both the sufficient and necessary condition is that $W_1 \subset W_2$.
Proof: Observe that $W_1 \subset W_2 \iff W_1 = (W_1 \cap W_2)$. Then the forward implication $W_1 \subset W_2 \Longrightarrow \dim(W_1) = \dim(W_1 \cap W_2)$ becomes a trivial statement. On the other hand we can consider the basis for the subspace $W_1 \cap W_2$ and we call it $\beta_\cap$. The we can extend this basis to a basis if W_1 and call it β_1. Then the statement: $\dim(W_1) = \dim(W_1 \cap W_2)$ is equivalent to the statement $|\beta_1| = |\beta_{cap}|$, therefore the extension was not necessary and hence $\beta_\cap = \beta_1$ which means that $W_1 = W_1 \cap W_2$.

Question 8: 1.6/26

Claim: The basis for this subspace is a set of polynomials $\{x-a, x^2-a^2, \dots, x^n - a^n\}$, and hence the dimension of this subspace is n.
Proof: First we will show linear independence:

$$c_1(x-a) + c_2(x^2-a^2) + \dots + c_n(x^n - a^n) = \sum c_i x^i - \sum c_i a^i$$

Now this polynomial has to be a zero polynomial for all x, in particular $x = 0$, substituting this in we get $\sum c_i a^i = 0$ then we get $\sum c_i x^i = 0$ which can be factored into x and we get $x(\sum c_i x^{i-1}) = 0$, which means that $c_{n-1} = 0$ and we can repeat this process to get $c_1 = c_2 = \dots = c_n = 0$. Then we can check for span, let a general polynomial on our subspace be of form $f(x) = c_0 + c_1 x + \dots + c_n x^n = \sum c_i x^i$, then I claim that $c_0 = -\sum c_i a^i$. This because $f(a) - c_0 = \sum c_i a^i = -c_0$, hence the equality. But any linear combination of our basis will have a form: $\sum c_i(x^i - a^i) = \sum c_i x^i - \sum c_i a^i$ Hence our basis spans the space.

Question 9: 1.6/31

part a)

We know that $W_1 \cap W_2$ forms a subspace and hence has a basis $\beta_\cap$. Since $W_1 \cap W_2 \subset W_1$ then this basis can be extended onto a basis of W_1. Hence $|\beta_1| \geq |\beta_\cap|$. Therefore $\dim(W_1) \geq \dim(W_1 \cap W_2)$.

part b)

Claim: Let the bases for W_1 and W_2 be β_1 and β_2 respectively. Then the $\beta_1 \cup \beta_2$ contains a basis for $W_1 + W_2$.
Proof: We just need to show that $\beta_1 \cup \beta_2$ spans $W_1 + W_2$. But since by definition $W_1 + W_2 = \{u + v, u \in W_1 \; v \in W_2\}$, then the span of $\beta_1 \cup \beta_2$ contains all the vectors of form $u + v$ where $u \in W_1$ and $v \in W_2$. Hence it spans and therefore the basis of $W_1 + W_2$ called $\beta_+ \subset \beta_1 \cup \beta_2$.
Therefore $|\beta_+| \leq |\beta_1 \cup \beta_2| \leq |\beta_1| + |\beta_2|$, hence $\dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2)$.

Question 10: 1.6/32

a) Example 1

Consider $W_1 = \text{span}((1,0,0))$, $W_2 = \text{span}((1,0,0),(0,1,0))$. Then $(W_1 \cap W_2) = W_1$ and the equality is trivially satisfied.

b) Example 2

Consider $W_1 = \text{span}((1,0,0),(0,1,0))$, $W_2 = \text{span}((0,0,1))$. Then $W_1 + W_2 = \mathbb{R}^3$ and from the bases $\dim(W_1) = 2$, whereas $\dim(W_2) = 2$, which means the equality is satisfied.

Question 11: 1.7/3

Suppose the set of real numbers is a vector space over rational numbers with a finite dimension n. Than any set of linearly independent vectors will have atmost n vectors. Consider the set of vectors $\{1, \pi, \pi^2, \ldots, \pi^n\}$. Then the span of this set is $\sum a_i \pi^i$, however, this expression cannot be equal to 0 with at least one of a_i's nonzero, because if it would then there would exist a polynomial of form $f(x) = \sum a_i x^i$ that has π as a solution, since π is transcendental. Hence, for any $n \in \mathbb{N}$ there exists a set of $n + 1$ linearly independent vectors in $\mathbb{R}$ and so it cannot be finite dimensional.