Problem Set 1 - Solutions

J. Scholtz

October 1, 2006

Question 1 (1.1/2a)

Let $\vec{a} = (3, -2, 4)$ and $\vec{b} = (-5, 7, 1)$. Then $\vec{b} - \vec{a} = (-8, 9, -3)$ and so the equation of the line is: $l : \vec{v} = (\vec{b} - \vec{a})t + \vec{a} = (-8, 9, -3)t + (3, -2, 4)$.

Question 2 (1.1/3a)

This is almost the same as previous question only in one more dimension: The can be parametrized as $P : \vec{v} = (\vec{b} - \vec{a})t + (\vec{c} - \vec{a})s + \vec{a}$. Plugging in the numbers gives us: $\vec{v} = (-2, 9, 7)t + (-5, 12, 2)s + (2, -5, -1)$.

Question 3 (1.2/8)

Given that $x, y \in V$ and $a, b \in F$ then we know we can use various Vectorspace and Field axioms to manipulate this expression. Since $a + b \in F$ then we can use VS7 to transform $(a + b)(x + y) = (a + b)x + (a + b)y$. Then by applying VS8 twice we get $ax + ay + bx + by$. Finally by VS1 we can rearange to get $ax + ay + bx + by$.

Question 4 (1.2/16)

If V is a set of all $m \times n$ matrices with real coefficients is V a vectorspace over $\mathbb{Q}$? We need to check whether all the axioms are satisfied. However, from the example 2 of the book we know that real matrices form a vectorspace over real numbers so we need to check a little less:
VS1, VS2, VS3, VS4 form an abelian group
VS5: $1 \in \mathbb{Q}$
VS6, VS7, VS8: Since $\mathbb{Q} \subset \mathbb{R}$ as a subfield then these axioms are satisfied.

Question 5 (1.2/19)

This construction does not form a vectorspace because it violates VS8. Consider: $(a + b)(x, y) = ((a + b)x, \frac{y}{a+b})$. However, by VS7 and VS8, $(a + b)(x, y) = a(x, y) + b(x, y) = (ax, y/a) + (bx, y/b) = ((a + b)x, \frac{y}{a} + \frac{y}{b}) \neq ((a + b)x, \frac{y}{a+b})$.

Question 6 (1.3/8 c,e)

In order to check that a set is a subspace we need to check whether it is closed under vector addition, scalar multiplication, existence of zero and additive inverses.

c) $W = \{(a_1, a_2, a_3), 2a_1 - 7a_2 + a_3 = 0\}$

$W = \{(a_1, a_2, a_3), 2a_1 - 7a_2 + a_3 = 0\}$ is a subspace. To make all the notation shorter we can note that if $\vec{z} = (2, -7, 1)$, then $\vec{v} \in V$ iff $\vec{z}.\vec{v} = 0$ (meaning standard dot product). Then it follows that if both $\vec{a}.\vec{z} = \vec{b}.\vec{z} = 0$ then also $(\vec{a} + \vec{b}).\vec{z} = 0$. Similarly if $\vec{a}.\vec{z} = 0$ then $(c\vec{a}).\vec{z} = c(\vec{a}.\vec{z}) = c.0 = 0$. Hence W is closed under vector addition and scalar multiplication. Note that $\vec{0} \in W$ and since we showed that this set is closed under scalar multiplication then it contains additive inverses as $(-1)\vec{v} = -\vec{v}$.

e) $W = \{(a_1, a_2, a_3), a_1 + 2a_2 - 3a_3 = 1\}$

This W is not a subspace. Consider two vectors $\vec{a} = (2, 1, 1)$ and $\vec{b} = (1, 3, 2)$. Then $\vec{a} + \vec{b} = (3, 4, 3)$ and for this vector $a_1 + 2a_2 - 3a_3 = 3 + 8 - 9 = 2 \neq 1$. Hence W is not closed under vector addition.

Question 7 (1.3/13)

Show that for any $s_0 \in S$ the set $L = \{f \in F(S, F) : f(s_0) = 0\}$ is a subspace of $F(S, F)$. Again we need to show that L is a subset of $F(S, F)$ and is closed under vector addition, scalar multiplication contains zero element and inverses. Let $f \in L$ and $g \in L$, then clearly $(f + g) \in L$ since $(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0$. Also if $c \in F$ then $cf \in L$ as $(cf)(s_0) = cf(s_0) = c.0 = 0$. Moreover, we know that the function $z(x) = 0, z(x) \in L$. Similarly for any function $f \in L$, let the additive inverse be $(-1)f \in L$.

Question 8 (1.3/16)

First, clearly C^n is a subset of $F(\mathbb{R}, \mathbb{R})$. This set is closed under addition since if both f, g have a continuous n-th derivative then the sum $(f + g)$ has a continuous n-th derivative. Also if $c \in F$ then cf will have a continuous n-th derivative since we are just multiplying by a fixed number. On top of that the zero function $f = 0$ is has any derivative equal to itself and it is continuous. Similarly since we have verified that we can multiply by scalars then the additive inverses are in C^n since if $f \in C^n$ then $-f \in C^n$.

Question 9 (1.3/23)

a)

First, we need to show that $W_1 + W_2$ is a subspace of W. It is closed under vector addition because if $u, v \in W_1 + W_2$, then there exist some $u_1, v_1 \in W_1$ and $u_2, v_2 \in W_2$ such that $u = u_1 + u_2$ and $v = v_1 + v_2$. Hence for any $u, v \in W_1 + W_2$, $u + v = u_1 + u_2 + v_1 + v_2 = (u_1 + v_1) + (u_2 + v_2) \in W_1 + W_2$. Similarly for scalar multiplication. Since W_1, W_2 are subspaces then they both contain $\vec{0}$ hence $W_1 + W_2$ contains 0. By closure of the scalar multiplication this set also contains all the additive inverses.
Second, we can show that both W_1 and W_2 are subspaces of $W_1 + W_2$. Since $W_1 + W_2$ contains all the vectors of form $w_1 + w_2$, $w_1 \in W_1$ and $w_2 \in W_2$. Then, first, let $w_1 = \vec{0}$ and we recover W_2, similarly set $w_2 = \vec{0}$ to recover W_1. In other words, we have shown that any vector $w_1 \in W_1$ is also present in $W_1 + W_2$ and hence $W_1 \subset W_1 + W_2$.

b)

Suppose that a subspace W contains both W_1 and W_2. Since W is a subspace it must be closed under addition, in particular it since it contains all $w_1 \in W_1$ and $w_2 \in W_2$ then it must contain all the vectors of form $w_1 + w_2$. Hence it contains $W_1 + W_2$.

Question 10 (1.3/28)

i) Show that skew symmetric matrices form a subspace:

Let M, N be skew symmetric matrices. Then $(M + N)^T = M^T + N^T = -M - N = -(M+N)$. Hence if M, N are skew symmetric so is $M+N$. Similarly if $c \in \mathbb{R}$ then $(cM)^T = cM^T = -(cM)$ therefore set is closed under vector addition and scalar multiplication. Note that the zero matrix satisfies the equation $0_{n\times n}^T = 0_{n\times n} = -0_{n\times n}$. Hence the set of skew symmetric matrices forms a subspace.

ii) Show that skew symmetric and symmetric add into the whole space

First we need to check that we can form a direct sum of these two spaces: that their intersection is the zero vector. Which matrix is both skew-symmetric and symmetric at the same time? In other words which matrix satisfies both properties: $M^T = M$ and $M^T = -M$? If both are true then $M = M^T = -M$, $M = -M$ which means that $M = 0_{n\times n}$. Therefore the intersection of symmetric and skew symmetric matrices is the zero matrix.
Take M to be any $n\times n$ matrix. Then let $S = (M+M^T)/2$ and $A = (M-M^T)/2$, (note: Since F does not have characteristic 2 we can divide by 2). First note that $S^T = S$ and $A^T = -A^T$. Secondly $S + A = M$. Therefore any matrix can be written as a sum of skew symmetric A and symmetric S. Hence the space of matrices is a direct sum of skew symmetric and symmetric matrices.

(Note: If you wonder what is field with charactersitic 2 and why the results should be different come to my or Thomas's office hours)

Question 11 (1.4/12)

In order to show that an "'if and only if"' statement is true, we need to show both ways, i.e. we need to show that if span(W) = W then W is a subspace and if W is a subspace then span(W) = W.

i) (span(W) = W) $\Rightarrow$ (W is a subspace)

The statement span(W) = W means that all the linear combinations of the form $\sum a_i \vec{v_i} \in W$. Hence the set W is closed under vector addition and scalar multiplication. Given this we know that it must also contain the additive identity and additive inverses. Hence W is a subspace.

ii) (span(W) = W) $\Leftarrow$ (W is a subspace)

The since W is a subspace then it must be closed under addition and scalar multiplication. In particular this means that $\sum a_i \vec{v_i} \in W$. But this means that all the linear combinations of any number of vectors in W are in W hence span(W) $\subset W$. Since we are taking all the vectors of W to form those linear combinations we can see that $W \subset$ span(W). Therefore span(W) = W.

Question 11 (1.4/15)

i) Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$

Note that by definition of intersection both S_1 and S_2 contain $S_1 \cap S_2$. Therefore by theorem 1.5, both $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1)$ and $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_2)$. From this follows that $\text{span}(S_1 \cap S_2) \subseteq (\text{span}(S_1) \cap \text{span}(S_2))$.

ii)

The trivial way how to satisfy the given equality is to let $S_1 = S_2$, since then the statement reduces to

$$\text{span}(S_1) = \text{span}(S_1)$$

On the other hand to get the inequality let $S_1 = \{(1,0);(0,1)\}$ and $S_2 = \{(2,0);(0,2)\}$. Then the intersection is clearly zero and the span is zero, whereas the intersection of both spans is the whole plain.