Solutions - Midterm II

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Question 1

1. Let us find the characteristic polynomial first since that gives us majority of the information we will need:

$$p(t) = \det(A - tI) = \begin{pmatrix} 1-t & 0 & 3 \\ 2 & 4-t & 2 \\ 3 & 2 & 1-t \end{pmatrix} = -t^3 + 6t^2 + 4t - 24 = -(t^3 - 6t^2 - 4t + 24)$$

   This means that $\det(A) = p(0) = -24$. Hence A is invertible and its rank then must be 3. We know that the eigenvalues are the roots of the characteristic polynomial. There are couple of ways to factorize it, but since we were given that the roots are integers then we could just find one by trial and error and divide the two polynomials. On the other hand from algebra we know that the second leading term of the polynomial gives us the negativ eof sume of roots and the constant term is their negative poduct in this case hence $abc = -24$ and $a + b + c = 6$, $a, b, c \in \mathbb{Z}$. The only solution to this is $a = 2$, $b = -2$ and $c = 6$ - the eigenvalues.
   The eigenvectors can be found as nonzero solutions to $(A - \lambda_i I)v = 0$ and we can get, $v_2 = (3, -4, 1)$, $v_{-2} = (-1, 0, 1)$ and $v_6 = (3, 8, 5)$.
   This means that the diagonal matrix and the change of basis matrix P are as follows:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$P = \begin{pmatrix} 3 & -1 & 3 \\ -4 & 0 & 8 \\ 1 & 1 & 5 \end{pmatrix}$$

   and we can check that $D = P^{-1}AP$.

2. This system can be written down as $x' = Ax$, hence $x' = PDP^{-1}x$ and chosing $y = P^{-1}x$, we get $y' = Dy$ and $x = Py$. But y can be solved quite easily $y = (C_1e^{2t}, C_2e^{-2t}, C_3e^{6t})$. Itt should be okay to leave the answer in the form: $x = P(C_1e^{2t}, C_2e^{-2t}, C_3e^{6t})$.

3. We can see that B is a trnasition matrix and that there will then be a stationary vector. Since $B = A/6$ then there will be only one eigenvalue equal to 1 and hence only one stationary eigenspace. By the theorem 5.20 in the book then the limit $\lim B^n = C$, where C is constructed by $(3, 8, 5)^t(1, 1, 1)/(3 + 8 + 5)$. Therefore:

$$limB^n = 1/16 \begin{pmatrix} 3 & 3 & 3 \\ 8 & 8 & 8 \\ 5 & 5 & 5 \end{pmatrix}$$

4. Since the previous matrix B is a transition matrix for this problem then:

$$P(\infty) = lim B^n(P_0) = 1/16 \begin{pmatrix} 3 & 3 & 3 \\ 8 & 8 & 8 \\ 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} 405 \\ 270 \\ 135 \end{pmatrix} = \begin{pmatrix} 151.875 \\ 405 \\ 253.125 \end{pmatrix}$$

Question 2

Question: Let V be a vector space with dimension n. Suppose $T : V \to V$ is a linear transformation with n distinct eigenvalues $a_1, \ldots, a_n$ and corresponding eigenvectors $v_1, \ldots, v_n$. Prove that the T-cyclic subspace W generated by $v = v_1 + v_2 + \ldots + v_n$ is equal to V.

Answer: Since W is a subspace of V it is sufficient to show it contains at least n linearly independent vectors. Suppose then that $v, Tv, T^2v \ldots, T^{n-1}v$ are linearly dependent. Then this means that for some, not all zero b_i's:

$$0 = \sum b_i T^i v = \sum_i b_i \sum_j a_j^i v_j = \sum_j \left(\sum_i b_i a_j^i \right) v_j$$

Since we know that v_j's are linearly independent (there are n different eigenvalues and so they form a basis for the sapce, hence they must be linearly independent), then this means that each term in the last sum must be zero on its own. Therefore we get:

$$\sum_i b_i a_j^i = 0 \ \forall j$$

This means that all the eigenvalues are solutions to the polynomial $\sum_i b_i t^i$. But this polynomial has only degree $n-1$, although there are n solutions to it (there are n eigenvalues) - Contradiction.