Midterm Solutions

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Question 1: True vs. False

1. **Question:**Let W_1, W_2 be subspaces of V and β_1, β_2 be their respective basis. Then is it true that $\beta_1 \cap \beta_2$ is a basis for $W_1 \cap W_2$?
   Answer: No, consider $W_1 = W_2 = \mathbb{R} \subset \mathbb{R}$. Let $\beta_1 = -1$ and $\beta_2 = 1$. Then $\beta_1 \cap \beta_2 = 0$ and so it is not a basis for $W_1 \cap W_2 = \mathbb{R}$.

2. **Question:** Let $P(\mathbb{R})$ denote the infinite dimensional vector space of all polynomials with real ceficients. Let $g(x) \in P(\mathbb{R})$ be some fixed polynomial. Then the function $T : P(\mathbb{R}) \to P(\mathbb{R})$, such that $T(f(x)) = g(x)f(x)$ is linear.
   Answer: Yes, we can check that $T(c \times a(x) + b(x)) = g(x)(c \times a(x) + b(x)) = g(x)(c \times a(x)) + g(x)b(x) = c \times T(a(x)) + T(b(x))$.

3. **Question:** Let V be a vector space and $T, U : V \to V$ be two linear operators. Then $N(T) \subset N(TU)$.
   Answer: No. Consider $V = \mathbb{R}^2$ and the maps $T(x, y) = (x, 0)$ and $U(x, y) = (y, x)$. Then $N(T) = \text{span}((0, 1))$, whereas $N(TU) = \text{span}((1, 0))$.

4. **Question:** Let V be a vector space and $T, U : V \to V$ be two linear operators. Then $R(TU) \subset R(T)$.
   Answer: Yes. $R(TU) = T(U(V))) = T(R(U)) \subset T(V)$ since $U(V) = R(U) \subset V$.

5. **Question:** Let S_1 and S_2 be two subsets of a vector space V, then let

   $$S = S_1 + S_2 = \{s_1 + s_2 | s_1 \in S_1, s_2 \in S_2\}$$

   Then consider if $\text{span}(S_1 \cup S_2) = \text{span}(S_1 + S_2)$.
   Answer: No. Let $S_1 = \{v_1\}$, $S_2 = \{v_2\}$, then $\text{span}(S_1 + S_2) = \{a(v_1 + v_2)\}$, whereas $\text{span}(S_1 \cup S_2) = bv_1 + cv_2$ hence these two cannot be equal.

Question 2: Dual Spaces

Let $V = P_3(\mathbb{R})$ and $\beta = 1, x, x^2, x^3$ be the basis for V. Then let $T : V \to V$ such that $T = 2\frac{d}{dx} - \frac{d^2}{dx^2}$. You may assume that T is linear.

1. **Question:** Compute the matrix $[T]_\beta$.
   Answer: We need to look at what the operator does to our basis. $T(1) = 0$, $T(x) = 2$, $T(x^2) = 4x - 2$, $T(x^3) = 6x^2 - 6x$. Hence the matrix will look like:

   $$[T]_\beta = \begin{pmatrix} 0 & 2 & -2 & 0 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. **Question:** Compute the matrix $[T^t]_{\beta^*}$.
   Answer: By theorem from our book we now that $[T^t]_{\beta^*} = ([T]_\beta)^t$. Therefore

$$[T^t]_{\beta^*} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & -6 & 6 & 0 \end{pmatrix}$$

3. **Question:** If $f \in V^*$ and $f(p) = p(2)$, then calculate $[f]_{\beta^*}$ and $T^t(f)$.
   Answer: Let us first see how f acts on our standard basis β. $f(ax^3+bx^2+cx+d) = 8a+4b+2c+d$. Therefore:

$$[f]_{\beta^*} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}$$

   There are two ways how to compute the value of $T^t(f)$ on a polynomial. The easier way is to realise that $T^t(f) = f(T)$, hence $T^t(f(p(x))) = f(T(p(x)))$, hence $(T^t(f))(p(x)) = \left(2\frac{dp(x)}{dx} - \frac{d^2p(x)}{dx^2}\right)_{x=2}$.

Question 3: Dimension Theorem

Let $V = M_{2\times 2}(\mathbb{R})$ denote the vector space of 2×2 matrices with real entries.

1. **Question:** Let W denote the set of matrices $S = \{A = (a)_{ij} \in V : \ a_{11} + a_{12} = a_{12}\}$. Show that W is a vector subspace of V
   Answer: First, observe that $0_{2\times 2} \in W$. Then let $A, B \in W$. Therefore: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{11} + a_{12} & a_{22} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{11} + b_{12} & b_{22} \end{pmatrix}$ and so $A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{11} + a_{12} + b_{11} + b_{12} & a_{22} + b_{22} \end{pmatrix} \in W$. Similarly for scalar multiplication.

2. **Question:** Let $U : V \to \mathbb{R}$ be a transformation defined as $U(A) = a_{21} - a_{12} - a_{11}$. Prove that U is linear.
   Answer: We can check that $U(A+B) = (a_{21}+b_{21}) - (a_{12}+b_{12}) - (a_{11}+b_{11}) = U(A) + U(B)$. Similarly $U(cA) = (ca_{21} - ca_{12} - ca_{11}) = c(a_{21} - a_{12} - a_{11}) = cU(A)$. Therefore this map is linear.

3. **Question:** State the Dimension Theorem
   Answer: Let $T : V \to W$ be a linear map, where both V, W are finite dimensional. Then

$$\dim(N(T)) + \dim(R(T)) = \dim(V)$$

4. **Question:** Use the Dimension Theorem to calculate the dimension of W.
   Answer: Observe that $N(T) = W$ and $R(T) = \mathbb{R}$, hence by the dimension theorem:

$$\begin{aligned} \dim(W) + \dim(\mathbb{R}) &= \dim(V) \\ \dim(W) + 1 &= 4 \\ \dim(W) &= 3 \end{aligned}$$

5. **Question:** Find a basis for W.
   Answer: Consider the set of matrices

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

We can quite easily check that $\beta \subset W$. Moreover since there are three matrices and $\dim(W) = 3$, we only need to check linear independence. Setting the span equal to the zero matrix yields:

$$\begin{pmatrix} a & b \\ a+b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Which implies $a = b = c = 0$, therefore this set is linearly independent.

The Bonus Question

Just shortly, if $T(T(v)) = T(v)$ for all $v \in V$, then this means that $T(T(v) - v) = 0$, hence $T(v) - v \in N(T)$, and so $T(v) - v = w$, where $w \in N(T)$, let $z = T(v)$, so since $T(v) \in R(T)$, then this means that for any $v \in V$, $z - v = w$ and after rearanging we get $v = z - w$. So every $v \in V$ can be expressed as a sum of a vector in $N(T)$ and $R(T)$. This implies that $V = R(T) + N(T)$. Now suppose that $v \in N(T) \cap R(T)$, then $T(v) = 0$ and $v = T(w)$. Therefore $TT(w) = T(v) = 0$, hence $N(T) \cap R(T) = \{0\}$. Done.
On the other hand the converse is not true. Consider $T : \mathbb{R} \to \mathbb{R}$ such that $T(x) = 2x$. Then $N(T) = 0$ and $R(T) = \mathbb{R}$, so clearly $\mathbb{R} = \mathbb{R} \oplus \{0\}$. But $T(T(x)) = 4x \neq 2x = T(x)$.